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# On anti-pentadiagonal persymmetric Hankel matrices with perturbed corners

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ABSTRACT

In this paper we express the eigenvalues of real anti-pentadiagonal persymmetric Hankel matrices with perturbed corners as the zeros of explicit rational functions. From these prescribed eigenvalues we give an orthogonal diagonalization for these matrices and a formula to compute its integer powers. In particular, an explicit expression not depending on any unknown parameter for the determinant and the inverse of complex anti-pentadiagonal persymmetric Hankel matrices with perturbed corners is provided.



## 1. Introduction

Latterly, many authors have considered the problem of calculating, so explicit as possible, the integer powers of anti-tridiagonal Hankel matrices (see, for instance, [1–9]). The main motivation for this issue comes from several areas of mathematics such as numerical analysis, differential equations, linear dynamical systems or graph theory where the computation of matrix powers is required.

One of the major purposes in this sequel is to establish an orthogonal diagonalization for real anti-pentadiagonal persymmetric Hankel matrices with perturbed corners. To achieve this goal, we shall employ a method found by Fasino in the late eighties called *modification technique* to obtain an orthogonal block diagonalization for these matrices, on one hand, and use well-known results about symmetric matrices modified by a rank-one matrix (developed by Golub and Bunch, Nielsen & Sorensen in the seventies) to locate the eigenvalues of real anti-pentadiagonal persymmetric Hankel matrices with perturbed corners and calculate its eigenvectors at the expense of those prescribed eigenvalues, on the other. Additionally, explicit formulae non-depending of any unknown parameter to compute the determinant and the inverse of complex anti-pentadiagonal persymmetric Hankel matrices with perturbed corners (assuming its nonsingularity) are made available.

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We say that an $n \times n$ matrix is *anti-pentadiagonal* if it has the form

$$\begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & \alpha_n & \beta_n & c_n \\ \vdots & & & & \iddots & \alpha_{n-1} & \beta_{n-1} & c_{n-1} & b_n \\ \vdots & & & \iddots & \iddots & \beta_{n-2} & c_{n-2} & b_{n-1} & a_n \\ \vdots & & \iddots & \iddots & \iddots & c_{n-3} & b_{n-2} & a_{n-1} & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & \alpha_4 & \beta_4 & c_4 & \iddots & \iddots & \iddots & & \vdots \\ \alpha_3 & \beta_3 & c_3 & b_4 & \iddots & \iddots & & & \vdots \\ \beta_2 & c_2 & b_3 & a_4 & \iddots & & & & \vdots \\ c_1 & b_2 & a_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Throughout, we shall consider the following $n \times n$ anti-pentadiagonal matrix

$$\mathbf{H}_n = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & a & b & r \\ \vdots & & & & \iddots & a & b & c & b \\ \vdots & & & \iddots & \iddots & b & c & b & a \\ \vdots & & \iddots & \iddots & \iddots & c & b & a & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & a & b & c & \iddots & \iddots & \iddots & & \vdots \\ a & b & c & b & \iddots & \iddots & & & \vdots \\ b & c & b & a & \iddots & & & & \vdots \\ r & b & a & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}. \tag{1.1}$$

## 2. An orthogonal diagonalization of the matrix $\mathbf{H}_n$

We begin this section presenting an auxiliary result concerning to the eigenvalues and eigenvectors of a diagonal matrix modified by a rank-one matrix which plays a central role along this paper. Let us point out that this problem was extensively studied in the past by some authors, particularly by Golub (see [10]) and Bunch, Nielsen & Sorensen (see [11]). Recall that in the computation of the eigensystem of a diagonal matrix modified by a rank-one matrix, we shall assume the distinctness of all eigenvalues of the diagonal matrix (i.e. of all entries of its main diagonal). Notwithstanding, if the diagonal matrix has multiple eigenvalues then deflation can be used just as in [11] (see pages 32 and 33) to eliminate them converting the original problem into another one where the eigenvalues are simple, thus ensuring that the hypothesis holds.

**Lemma 1.** *Let $n \in \mathbb{N}$, $a, b, c, r \in \mathbb{R}$ such that $c \neq a + r$ and*

$$\lambda_k := -2a\cos\left[\frac{(n-1)k\pi}{n+1}\right] - 2b\cos\left(\frac{nk\pi}{n+1}\right) - c\cos(k\pi), \quad k = 1, \ldots, n. \tag{2.1}$$

(a) *If $n$ is even,*

$$\mathbf{u} = \begin{bmatrix} \frac{2}{\sqrt{n+1}}\sin\left(\frac{\pi}{n+1}\right) \\ \frac{2}{\sqrt{n+1}}\sin\left(\frac{3\pi}{n+1}\right) \\ \vdots \\ \frac{2}{\sqrt{n+1}}\sin\left[\frac{(n-1)\pi}{n+1}\right] \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} \frac{2}{\sqrt{n+1}}\sin\left(\frac{2\pi}{n+1}\right) \\ \frac{2}{\sqrt{n+1}}\sin\left(\frac{4\pi}{n+1}\right) \\ \vdots \\ \frac{2}{\sqrt{n+1}}\sin\left(\frac{n\pi}{n+1}\right) \end{bmatrix} \tag{2.2a}$$