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Numerical study of three-dimensional Turing patterns using a meshless method based on moving Kriging element free Galerkin (EFG) approach

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ABSTRACT

In this paper a numerical procedure is presented for solving a class of three-dimensional Turing system. First, we discrete the spatial direction using element free Galerkin (EFG) method based on the shape functions of moving Kriging interpolation. Then, to achieve a high-order accuracy, we use the fourth-order exponential time differencing Runge–Kutta (ETDRK4) method. Using this discretization for the temporal dimension, we obtain an explicit scheme and do not need to solve nonlinear system of equations. The EFG method uses a weak form of the considered equation that is similar to the finite element method with the difference that in the EFG method test and trial functions are moving least squares approximation (MLS) shape functions. Since the shape functions of moving least squares (MLS) approximation do not have Kronecker delta property, we cannot implement the essential boundary condition, directly. Also building shape functions of MLS approximation is a time consuming procedure. Because of the mentioned reasons we employ the shape functions of moving Kriging interpolation technique which have the mentioned property and less CPU time is required for building them. For testing this method on three-dimensional PDEs, we select some equations and system of PDEs such as Allen–Cahn, Gray–Scott, Ginzburg–Landau, Brusselator models, predator–prey model with additional food supply to predator. Several test problems are solved and numerical simulations are reported which confirm the efficiency of the proposed scheme.

1. Introduction

The partial differential equations (PDEs) can be used for explaining many phenomena in physics, biology, chemistry, engineering and social sciences [1]. There are two general categories of PDEs i.e. linear and nonlinear equations in which many physical phenomena have been modeled by nonlinear PDEs. In other hand, there are two basic strategies for solving PDEs i.e. analytical and numerical methods. In the meantime, the analytical methods provide a closed form for the solution of nonlinear PDEs. But finding a solution to closed form always is not possible, especially when there are many nonlinear terms in PDEs. For this reason, numerical methods are useful for finding an approximation solution for PDEs. In this paper, we solve some problems which have interesting applications in biological sciences such as system of equations for Turing patterns, Ginzburg–Landau model and Allen–Cahn equation. Turing systems appear in various biological systems, such as

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patterns in fish, butterflies, lady bugs and etc. [2]. The main aim of this paper is to propose an efficient scheme for solving three-dimensional equation with high-order accuracy in time-direction based on the explicit fourth-order technique.

In this paper, we consider the class of systems of equations to following form

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v) + F(u, v), \\ \dfrac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v) + G(u, v), \end{cases} \quad (x, y) \in \Omega, \ t > 0, \tag{1.1}$$

with initial conditions

$$u(x, y, 0) = u_0(x, y), \qquad v(x, y, 0) = v_0(x, y), \quad (x, y) \in \Omega, \tag{1.2}$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, \quad (x, y) \in \partial\Omega, \ t > 0, \tag{1.3}$$

or periodic boundary conditions, where $f(u, v)$ and $g(u, v)$ are linear functions respect to u and v, also $F(u, v)$ and $G(u, v)$ are nonlinear functions respect to u and v and ∇^2 is operator of Laplacian. First, we would like to mention that the models investigated in the current paper are:

1. **Gray–Scott model**
2. **Allen–Cahn model**
3. **Ginzburg–Landau model**
4. **Brusselator model**
5. **predator–prey model with additional food supply to predator**

1.1. Complex Ginzburg–Landau model

The present subsection is taken from wikipedia site.[1] The Ginzburg–Landau is a theory in physics that has been named because of Vitaly Lazarevich Ginzburg and Lev Landau. The Ginzburg–Landau theory is a theory in mathematical physics used to describe superconductivity. In its initial form, it was postulated as a phenomenological model which could describe type-I superconductors without examining their microscopic properties. Later, a version of Ginzburg–Landau theory was derived from the Bardeen–Cooper–Schrieffer microscopic theory by Lev Goŕkov, thus showing that it also appears in some limit of microscopic theory and giving microscopic interpretation of all its parameters [3].

Based on Landau's previously-established theory of second-order phase transitions, Ginzburg and Landau argued that the free energy, F, of a superconductor near the superconducting transition can be expressed in terms of a complex order parameter field, $|\psi|$, which is nonzero below a phase transition into a superconducting state and is related to the density of the superconducting component, although no direct interpretation of this parameter was given in the original paper. Assuming smallness of $|\psi|$ and smallness of its gradients, the free energy has [3] the form of a field theory i.e.

$$F = F_n + \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4 + \frac{1}{2m}\,|(-i\hbar\nabla - 2e\mathbf{A})\,\psi|^2 + \frac{|\mathbf{B}|^2}{2\mu_0}, \tag{1.4}$$

in which

- F_n is the free energy in the normal phase
- α and β in the initial argument are treated as phenomenological parameters
- m is an effective mass
- e is the charge of an electron
- $\mathbf{A}$ is the magnetic vector potential
- $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field.

By minimizing the free energy with respect to variations in the order parameter and the vector potential, one arrives at the Ginzburg–Landau equations [3]

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}\,(-i\hbar\nabla - 2e\mathbf{A})^2\,\psi = 0, \tag{1.5}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j}; \quad \mathbf{j} = \frac{2e}{m}\mathrm{Re}\left\{\psi^*\,(-i\hbar\nabla - 2e\mathbf{A})\,\psi\right\}, \tag{1.6}$$

where j denotes the dissipation-less electrical current density and *Re* is the real part [3]. As is said in [4] the Ginzburg–Landau equation has been used as a mathematical model for various pattern formation systems in mechanics, physics and chemistry.

[1] https://en.wikipedia.org/wiki/Ginzburg-Landau-theory.