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Discontinuous Galerkin method for the solution of a transport level-set problem☆

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ABSTRACT

The subject of the paper is the numerical analysis of the transport level-set problem discretized by the discontinuous Galerkin method. Without the assumption that the first order nonstationary transport equation contains a reaction term, which is used in a standard literature, we prove error estimates in the $L^\infty(L^2)$-norm in the case of the space semidiscretization method of lines and in the case of the space–time discontinuous Galerkin method in the $L^2(L^2)$-norm. Numerical experiments support the derived error estimates and show that they are not sharp in the case of the space–time discontinuous Galerkin method.

1. Introduction

In a number of problems of science and technology it is necessary to solve various types of convection–diffusion problems. We can mention, for example, fluid dynamics, heat and mass transfer, environmental protection, water transfer in soils, porous media flow, but also financial mathematics and image processing. Therefore, developing efficient numerical methods for numerical solving convection–diffusion problems represents an important part of the numerical solution of partial differential equations. There is an extensive literature devoted mainly to linear convection–diffusion problems, represented by the monographs [1–4]. The solution of nonlinear convection–diffusion equations is treated in the monograph [5], where the discontinuous Galerkin method (DGM) is used.

In various applications we also meet the necessity to analyze numerically linear convection transport problems

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi = 0, \tag{1}$$

where $\varphi(x,t)$ is the unknown solution, $\boldsymbol{v}(x,t)$ is the prescribed flow velocity function, ∂_t is the partial derivative with respect to time t and ∇ is the gradient operator. Among other, it is the case of the level-set method applied, for example, to the simulation of multiphase flow, crystal growth, computational geometry, image processing and material science. See, e.g. [6–12]. In these works, relatively simple finite difference, finite volume or conforming finite element methods are used. It is important to be able to treat numerically problems with the transport of discontinuities, which usually causes difficulties manifested either by smearing the discontinuities due to low order of applied methods or spurious oscillations in the case

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of higher order methods. A promising technique for avoiding these difficulties is the discontinuous Galerkin method (DGM) using piecewise polynomial in general discontinuous approximations. There is an extensive literature concerned with this method, whose amount is rapidly increasing. We can refer the reader to the monographs [13,5,14] and references therein. As for the DGM for linear convection–diffusion problems, we mention the pioneering works [15,16], where stationary problem was analyzed. Theoretical analysis of a nonstationary problem by the space DGM semidiscretization (i.e., the method of lines) is the subject of [17].

The numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of high order of accuracy both in space and in time. For some applications, the standard Euler schemes or θ-schemes [18,19] are not sufficiently accurate in time. In computational fluid dynamics, Runge–Kutta methods are very popular. Let us mention, for example, the well-known Runge–Kutta discontinuous Galerkin methods (see e.g. [20]). They are applicable to the numerical solution of a wide class of problems, including nonlinear conservation laws and nonlinear convection–diffusion problems, but they are conditionally stable. An example of unconditionally stable method is the technique using the backward difference formula (BDF). It was used for the solution of compressible flow, e.g. in [21] and analyzed theoretically in the case of a scalar nonlinear convection–diffusion equation in [22]. However, the stability of these methods is limited, particularly in the case of nonsymmetric problems. Moreover, the change of the length of time steps is not easy. In [23], a time discretization of arbitrary order of parabolic problems was proposed and analyzed. Unfortunately, it is applicable to linear problems only. All the mentioned methods of time discretization require to use identical space meshes.

One technique which does not suffer from the mentioned difficulties is the discontinuous Galerkin time discretization. This technique was introduced and analyzed, e.g. in [16] for the solution of ordinary differential equations. In [24–29], the solution of parabolic problems is carried out with the aid of conforming finite elements in space combined with the DG time discretization. We can also refer to the monograph [30], where the discontinuous Galerkin time discretization is applied to abstract parabolic operator equations in Hilbert spaces.

The DG time discretization can also be combined with the space discontinuous Galerkin discretization. In this way we can obtain an unconditionally stable space–time discontinuous Galerkin method (STDGM) for the numerical solution of nonstationary, in general nonlinear initial–boundary value problems. The further advantage of the STDGM is a natural treatment of different space-discretization on different time levels which is important in adaptive time-dependent simulations.

The analysis of the STDGM applied to the advection–diffusion and the Oseen problems was presented in [31,32], respectively. The application of the STDGM to fluid dynamics problems was given in [33–36]. In [31,32], the time is treated as an additional variable and thus the problem (1) is re-written as

$$\boldsymbol{w} \cdot \widetilde{\nabla}\varphi = 0, \tag{2}$$

where $\boldsymbol{w} = (1, \boldsymbol{v}) \in \mathbb{R}^{d+1}$ (d is the space dimension) is the space–time advection vector and $\widetilde{\nabla}\varphi = (\partial_t\varphi, \partial_{x_1}\varphi, \dots, \partial_{x_d}\varphi)$ is the space–time gradient operator. Using this approach more general $(d+1)$-dimensional meshes are allowed.

In this paper we employ a different technique, similar to [24,16,27,29,30] cited above, where the time variable is still treated differently from the space variables. We combine the time DG discretization with the space DG semi-discretization. This approach was used in [37,38] for the solution of convection–diffusion problems. The time discretization is considered fully implicitly, which does not allow the local (in space) time-stepping in contrast to [33–35]. On the other hand, the resulting scheme is practically unconditionally stable which can be advantageous in the choice of the size of the time steps.

In the theoretical analysis presented in the above works an important assumption is the combination of the convection and diffusion with a linear reaction term cu satisfying the condition $c - \operatorname{div} \boldsymbol{v}/2 \geq \gamma > 0$, where c is the reaction coefficient, $\boldsymbol{v}$ is the transport velocity and γ is a constant. This condition (which is also applied in the framework of stabilized conforming finite elements analyzed, e.g., in [3]) allows to derive error estimates uniform with respect to the diffusion coefficient. However, in some problems this condition is not satisfied, because the diffusion and reaction vanish. This is the case of the level-set problem.

In papers by Cockburn and Shu [39,40], the DGM was analyzed for a conservation law

$$\partial_t\varphi + \nabla \cdot (\boldsymbol{v}\varphi) = 0, \tag{3}$$

(more precisely for its nonlinear variant). However the time discretization was carried out by explicit Runge–Kutta method. Let us note that (3) is equivalent to (1) only for the case when $\nabla \cdot \boldsymbol{v} = 0$. Up to our knowledge, the presented analysis of the STDG discretization of a convection transport equation without diffusion and reaction terms is original and completely new. The presented STDG scheme completely differs from [41,33], where the local (explicit) time stepping was used. There it is not necessary to solve a large linear algebraic system however, the resulting scheme has to be conditionally stable with respect to the size of the time step.

In this paper we complete the theory of the DGM applied to the nonstationary transport level-set equation. Section 2 introduces some notation and formulates the continuous problem. Moreover, the space and time DG discretizations are derived here. Section 3 is devoted to the error estimation for the method of lines and for the full STDGM. In Section 4 some numerical experiments are presented demonstrating the derived error estimates.