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Breaking spaces and forms for the DPG method and applications including Maxwell equations[☆]

C. Carstensen [a], L. Demkowicz [b], J. Gopalakrishnan [c,*]

[a] Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany
[b] Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX 78712, USA
[c] PO Box 751, Portland State University, Portland, OR 97207-0751, USA

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ABSTRACT

Discontinuous Petrov–Galerkin (DPG) methods are made easily implementable using "broken" test spaces, i.e., spaces of functions with no continuity constraints across mesh element interfaces. Broken spaces derivable from a standard exact sequence of first order (unbroken) Sobolev spaces are of particular interest. A characterization of interface spaces that connect the broken spaces to their unbroken counterparts is provided. Stability of certain formulations using the broken spaces can be derived from the stability of analogues that use unbroken spaces. This technique is used to provide a complete error analysis of DPG methods for Maxwell equations with perfect electric boundary conditions. The technique also permits considerable simplifications of previous analyses of DPG methods for other equations. Reliability and efficiency estimates for an error indicator also follow. Finally, the equivalence of stability for various formulations of the same Maxwell problem is proved, including the strong form, the ultraweak form, and various forms in between.

1. Introduction

When a domain Ω is partitioned into elements, a function in a Sobolev space like $H(\mathrm{curl}, \Omega)$ or $H(\mathrm{div}, \Omega)$ has continuity constraints across element interfaces, e.g., the former has tangential continuity, while the latter has continuity of its normal component. If these continuity constraints are removed from the space, then we obtain "broken" Sobolev spaces. Discontinuous Petrov–Galerkin (DPG) methods introduced in [1,2] used spaces of such discontinuous functions in broken Sobolev spaces to localize certain computations. The studies in this paper begin by clarifying this process of breaking Sobolev spaces. This process, sometimes called hybridization, has been well studied within a discrete setting. For instance, the hybridized Raviart–Thomas method [3,4] is obtained by discretizing a variational formulation and then removing the continuity constraints of the discrete space, i.e., by discretizing first and then hybridizing. In contrast, in this paper, we identify methods obtained by hybridizing first and then discretizing, a setting more natural for DPG methods. We then take this idea further by connecting the stability of formulations with broken spaces and unbroken spaces, leading to the first convergence proof of a DPG method for Maxwell equations.

Section 2 is devoted to a study of the interface spaces that arise when breaking Sobolev spaces. These infinite-dimensional interface spaces can be used to connect the broken and the unbroken spaces. The main result of Section 2, contained in Theorem 2.3, makes this connection precise and provides an elementary characterization (by duality) of the natural norms on these interface spaces. This theorem can be viewed as a generalization of a similar result in [5].

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* Corresponding author. Tel.: +1 503 725 2205.
E-mail addresses: cc@math.hu-berlin.de (C. Carstensen), leszek@ices.utexas.edu (L. Demkowicz), gjay@pdx.edu (J. Gopalakrishnan).

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Having discussed breaking spaces, we proceed to break variational formulations in Section 3. The motivation for the theory in that section is that some variational formulations set in broken spaces have another closely related variational formulation set in their unbroken counterpart. This is the case with all the formulations on which the DPG method is based. The main observation of Section 3 is a simple result (Theorem 3.3) which in its abstract form seems to be already known in other studies [6]. In the DPG context, it provides sufficient conditions under which *stability of broken forms follows from stability of their unbroken relatives*. As a consequence of this observation, we are able to simplify many previous analyses of DPG methods. The content of Sections 2 and 3 can be understood without reference to the DPG method.

A quick introduction to the DPG method is given in Section 4, where known conditions needed for *a priori* and *a posteriori* error analysis are also presented. One of the conditions is the existence of a Fortin operator. Anticipating the needs of the Maxwell application, we then present, in Section 5, a sequence of Fortin operators for $H^1(K)$, $H(\text{curl}, K)$ and $H(\text{div}, K)$, all on a single tetrahedral mesh element K. They are constructed to satisfy certain moment conditions required for analysis of DPG methods. They fit into a commuting diagram that helps us prove the required norm estimates (see Theorem 5.1).

The time-harmonic Maxwell equations within a cavity are considered afterward in Section 6. Focusing first on a simple DPG method for Maxwell equation, called the primal DPG method, we provide a complete analysis using the tools developed in the previous section. To understand one of the novelties here, recall that the wellposedness of the Maxwell equations is guaranteed as soon as the excitation frequency of the harmonic wave is different from a cavity resonance. However, this wellposedness is not directly inherited by most standard discretizations, which are often known to be stable solely in an asymptotic regime [7]. The discrete spaces used must be sufficiently fine before one can even guarantee solvability of the discrete system, not to mention error guarantees. Furthermore, the analysis of the standard finite element method does not clarify how fine the mesh needs to be to ensure that the stable regime is reached. In contrast, the DPG schemes, having inherited their stability from the exact equations, are stable no matter how coarse the mesh is. This advantage is striking when attempting robust adaptive meshing strategies.

Another focus of Section 6 is the understanding of a proliferation of formulations for the Maxwell boundary value problem. One may decide to treat individual equations of the Maxwell system differently, e.g., one equation may be imposed strongly, while another may be imposed weakly via integration by parts. Mixed methods make a particular choice, while primal methods make a different choice. We will show (see Theorem 6.3) that the stability of one formulation implies the stability of five others. The proof is an interesting application of the closed range theorem. However, when the DPG methodology is applied to discretize these formulations, the numerical results reported in Section 7, show that the various methods do exhibit differences. This is because the functional settings are different for different formulations, i.e., convergence to the solution occurs in different norms. Section 7 also provides results from numerical investigations on issues where the theory is currently silent.

2. Breaking Sobolev spaces

In this section, we discuss precisely what we mean by breaking Sobolev spaces using a mesh. We will define *broken spaces* and *interface spaces* and prove a duality result that clarifies the interplay between these spaces. We work with infinite-dimensional (but mesh-dependent) spaces on an open bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary. The mesh, denoted by Ω_h, is a disjoint partitioning of Ω into open elements K such that the union of their closures is the closure of Ω. The collection of element boundaries ∂K for all $K \in \Omega_h$, is denoted by $\partial\Omega_h$. We assume that each element boundary ∂K is Lipschitz. The shape of the elements is otherwise arbitrary for now.

We focus on the most commonly occurring first order Sobolev spaces of real or complex-valued functions, namely $H^1(\Omega)$, $H(\text{div}, \Omega)$, and $H(\text{curl}, \Omega)$. Their *broken* versions are defined, respectively, by

$$\begin{aligned}
H^1(\Omega_h) &= \{u \in L^2(\Omega) : u|_K \in H^1(K),\ K \in \Omega_h\} &&= \prod_{K\in\Omega_h} H^1(K),\\
H(\text{curl}, \Omega_h) &= \{E \in (L^2(\Omega))^3 : E|_K \in H(\text{curl}, K),\ K \in \Omega_h\} &&= \prod_{K\in\Omega_h} H(\text{curl}, K),\\
H(\text{div}, \Omega_h) &= \{\sigma \in (L^2(\Omega))^3 : \sigma|_K \in H(\text{div}, K),\ K \in \Omega_h\} &&= \prod_{K\in\Omega_h} H(\text{div}, K).
\end{aligned}$$

As these broken spaces contain functions with no continuity requirements at element interfaces, their discretization is easier than that of globally conforming spaces.

To recover the original Sobolev spaces from these broken spaces, we need traces and interface variables. First, let us consider these traces on each element K in Ω_h.

$$\begin{aligned}
\text{tr}^K_{\text{grad}} u &= u|_{\partial K} && u \in H^1(K),\\
\text{tr}^K_{\text{curl},\top} E &= (n_K \times E) \times n_K|_{\partial K} && E \in H(\text{curl}, K),\\
\text{tr}^K_{\text{curl},\dashv} E &= n_K \times E|_{\partial K} && E \in H(\text{curl}, K),\\
\text{tr}^K_{\text{div}} \sigma &= \sigma|_{\partial K} \cdot n_K && \sigma \in H(\text{div}, K).
\end{aligned}$$