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The steady states of a non-cooperative model arising in reactor dynamics

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ABSTRACT

In this work, we investigate the existence and stability of coexistence states of a reaction–diffusion model originated by the theory of nuclear reactors. By converting it to an equivalent system with linear cooperative part, we obtain the necessary and sufficient conditions for the existence of coexistence states via lower and upper solutions method. In addition, stability results and a criterion are also given to enrich and complement those available in the literature.

1. Introduction

This paper studies the existence and stability of positive solutions for the following non-cooperative reactor-diffusion steady state model of nuclear reactors

$$\begin{cases} -\Delta u = au - buv, \quad x \in \Omega, \\ -\Delta v = cu - duv - ev, \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}}, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$, and the coefficients $a,\ b,\ c,\ d$ and e take positive values. A pair (u, v) is called a positive solution of (1.1) if both u and v are positive in $\bar{\Omega}$.

It is well known that the parabolic system

$$\begin{cases} u_t - \Delta u = au - buv, \quad x \in \Omega,\ t > 0, \\ v_t - \Delta v = cu - duv - ev, \quad x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}}, \quad x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0 \succ 0, \qquad v(x, 0) = v_0 \succ 0, \quad x \in \Omega, \end{cases} \tag{1.2}$$

which corresponds to system (1.1), is a refinement of the prototype model in nuclear engineering proposed by Kastenberg and Chambré [1], by adding the diffusion and the nonlinear feedback to the temperature in [2]. Here $u \succ 0$ means $u \geq 0$ but

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$u \not\equiv 0$. In real world applications, the unknown functions u and v respectively stands for the density of fast neutrons and the temperature in the reactor. The homogeneous Neumann boundary condition means the boundary of the closed container is heat insulation and neutron flux cannot go through the boundary $\partial\Omega$.

In the past few decades, under several kinds of boundary conditions such as Dirichlet, Neumann and Robin type, above models and their variants had been of great interest for many researchers. We refer the readers here to [3–8,1,9–22] and the references therein for some research works on this topic. Meanwhile, some authors also paid their attention to the one-dimensional analogues of system (1.1), see Antón and López-Gómez [3,9], Chen [23,24], Chen and Ma [25,26], Li et al. [27] and Wang and An [28]. In particular, the models have received extensively study under the homogeneous Dirichlet boundary condition in most of above mentioned works, and the coefficient a is usually viewed as a bifurcation parameter. For instance, Arioli [2] provided the existence results for nontrivial periodic solutions and a global attractor, López-Gómez [9] was mainly concerned with the corresponding steady state problem, and in [3], the authors investigated the spatially heterogeneous counterpart of the model. Peng etc. [14], Zhou etc. [15] and Zhou [16] completed and sharpened those derived in [3,2,9].

In the present work, we are mainly concerned with the existence and stability of positive solutions of (1.1). The coefficients $a,\ b,\ c,\ d$ and e will be considered as parameters, and an approach which is completely different from those in above literatures will be taken. More precisely, system (1.1) will be converted to an equivalent system as below by setting $w = u - \frac{b}{d}v$,

$$\begin{cases} -\Delta w = \left(a - \dfrac{bc}{d}\right)w + \dfrac{b}{d}\left(a + e - \dfrac{bc}{d}\right)v, & x \in \Omega, \\ -\Delta v = cw + \left(\dfrac{bc}{d} - e\right)v - dwv - bv^2, & x \in \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}}, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Obviously, a positive solution (w, v) of system (1.3) yields a positive solution (u, v) of the original problem (1.1). Suppose that

(H1) $\frac{bc}{d} - e < a < \frac{bc}{d}$.

Then $\frac{b}{d}\left(a + e - \frac{bc}{d}\right) > 0$, and thus the matrix

$$A := \begin{pmatrix} a - \dfrac{bc}{d} & \dfrac{b}{d}\left(a + e - \dfrac{bc}{d}\right) \\ c & \dfrac{bc}{d} - e \end{pmatrix}$$

is a cooperative one, which also means the linear system

$$\begin{cases} -\Delta w = \left(a - \dfrac{bc}{d}\right)w + \dfrac{b}{d}\left(a + e - \dfrac{bc}{d}\right)v, & x \in \Omega, \\ -\Delta v = cw + \left(\dfrac{bc}{d} - e\right)v, & x \in \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}}, & x \in \partial\Omega \end{cases} \tag{1.4}$$

is cooperative. It is well known that cooperative systems (i.e., systems like (1.1) where the right-hand sides are nondecreasing in the off-diagonal terms) can be analyzed by applying lower and upper solutions method [9] and, in general, possess many of the properties of scalar semilinear equations. Note that although system (1.3) is not cooperative, its linear part (1.4) is and the cooperative nature of system (1.4) plays a key role in formulating and helping to prove our existence and stability results for system (1.1).

For simplicity, let $U = (w, v)^T, \mathscr{L} = \mathrm{diag}(-\Delta, -\Delta)$ and $F(U) = \binom{0}{dwv + bv^2}$. Then systems (1.3) and (1.4) can be equivalently rewritten as

$$\begin{cases} \mathscr{L}U = AU - F(U), & x \in \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0, & x \in \partial\Omega \end{cases} \tag{1.5}$$

and

$$\begin{cases} \mathscr{L}U = AU, & x \in \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$