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Two new variants of the HSS preconditioner for regularized saddle point problems ☆

Zhao-Zheng Liang, Guo-Feng Zhang *

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

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ABSTRACT

Two new preconditioners, which can be viewed as improved variants of the Hermitian and skew-Hermitian splitting (HSS) preconditioner, are presented for regularized saddle point problems. The unconditionally convergent properties of the corresponding iteration methods are deduced and the selection of optimal iteration parameters resulting in fast convergence for the two iteration methods are discussed. Moreover, eigenvalue bounds of the preconditioned matrices and upper bounds of the degree of their minimal polynomials are obtained. Compared with the HSS preconditioner, they are much better approximations to the coefficient matrix of the saddle point problem and they have better convergence properties and spectrum distributions. Numerical experiments from the Stokes problem are presented, which illustrates the advantages of the two preconditioners over the HSS preconditioner and some other preconditioners to accelerate the convergence rate of GMRES.

1. Introduction

We consider the regularized saddle point problems of the following structure:

$$\mathscr{A}_+ u \equiv \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv c_+, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a matrix of full row rank with $m < n$, $C \in \mathbb{R}^{m\times m}$ is a symmetric positive semidefinite matrix, and $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$ are given vectors.

The linear systems (1.1) arise in many scientific computing and engineering applications, such as computational fluid dynamics, mixed finite element approximation of elliptic PDEs, weighted and equality constrained least squares estimation, and so forth; see [1] and the references therein. In practice, C is often diagonal and its entries are small. Using interior point methods for constrained optimization, the entries in C generally become small as the optimization iteration process. For the Stokes problems, the entries of C are generally small since they scale with the underlying mesh size and so reduce for finer grids. See [2] for more discussions.

Recently, a large amount of efficient iteration methods have been proposed to solve the linear systems (1.1), such as inexact and parameterized Uzawa schemes [3,4], Krylov subspace methods [5,6], and so on. The Krylov subspace methods for regularized saddle point problems (1.1) are most often used in combinations with suitable approximation matrices for the

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* Corresponding author.
E-mail addresses: liangzhzh2014@lzu.edu.cn (Z.-Z. Liang), gf_zhang@lzu.edu.cn (G.-F. Zhang).

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coefficient matrix $\mathscr{A}_+$, which are called the preconditioners. With suitable preconditioners, the Krylov subspace methods can result in more rapid convergence than the former iteration methods. Numerous preconditioning techniques have appeared in the literature, including the block diagonal and block triangular preconditioners [7–15], constraint preconditioners [2,16–18], augmented Lagrangian preconditioners [19–24], and so on.

Negativing the second block row of (1.1) equivalently yields the nonsymmetric saddle point problems

$$\mathscr{A}u \equiv \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv c. \tag{1.2}$$

The nonsymmetric formulation (1.2) has certain desirable properties. We may lose symmetry, but we gain positive semi-definiteness [25]. This can be advantageous when using certain Krylov subspace methods, like restarted GMRES; see [26]. At the same time, under mild conditions, the conjugate gradient method in a nonstandard inner product can be used; see [27–29]. Besides, some special preconditioning techniques can be employed by taking advantages of the formulation (1.2).

The coefficient matrix $\mathscr{A}$ in (1.2) has the Hermitian and skew-Hermitian splitting (HSS) [30]:

$$\mathscr{A} = \begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix} \equiv \mathscr{H} + \mathscr{S}. \tag{1.3}$$

Given $\alpha > 0$, based on the splitting in (1.3), we have the following two splittings:

$$\mathscr{A} = (\alpha I + \mathscr{H}) - (\alpha I - \mathscr{S}) \quad \text{and} \quad \mathscr{A} = (\alpha I + \mathscr{S}) - (\alpha I - \mathscr{H}),$$

where I denotes the identity matrix of proper size. Then we have the following HSS iteration scheme

$$\begin{cases} (\alpha I + \mathscr{H})u^{\left(k+\frac{1}{2}\right)} = (\alpha I - \mathscr{S})u^{(k)} + c, \\ (\alpha I + \mathscr{S})u^{(k+1)} = (\alpha I - \mathscr{H})u^{\left(k+\frac{1}{2}\right)} + c. \end{cases} \tag{1.4}$$

The iteration scheme (1.4) can result from the splitting $\mathscr{A} = \mathscr{P}_{\text{HSS}} - (\mathscr{P}_{\text{HSS}} - \mathscr{A})$ with the HSS preconditioner

$$\mathscr{P}_{\text{HSS}} = \frac{1}{2\alpha}(\alpha I + \mathscr{H})(\alpha I + \mathscr{S}) = \frac{1}{2\alpha} \begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I + C \end{pmatrix} \begin{pmatrix} \alpha I & B^T \\ -B & \alpha I \end{pmatrix}. \tag{1.5}$$

In [25], convergence analysis of the HSS iteration scheme (1.4) and some spectral properties of the preconditioned matrix $\mathscr{P}_{\text{HSS}}^{-1}\mathscr{A}$ are established. Moreover, eigenvalue estimates of the preconditioned matrix $\mathscr{P}_{\text{HSS}}^{-1}\mathscr{A}$ are discussed in greater detail in [31,32]. Other works related to the HSS preconditioners include [27,33,34] for the case $C = 0$ and [35–37] for the case $C = \eta I$ with $\eta > 0$, which arises from weighted Toeplitz least squares problems. Besides, some preconditioning and relaxed variants of the HSS iteration scheme are studied in [38–42]. The HSS preconditioner is parameter dependent, so the selection of optimal parameters is crucial for its implementation. However, it is a tough thing for the selection of optimal iteration parameter. Optimization of the HSS iteration is realized in [33] for the case $C = 0$ by Fourier transforms for simple model problem of the Poisson equation. In [37] for the case $C = \eta I$ with $\eta > 0$, the optimal iteration parameter is also determined. To our knowledge, there are, however, no optimization results when C is a general nonzero matrix.

In this paper, two variants of the HSS preconditioner (1.5) are proposed for the regularized saddle point problems (1.2). They are much better approximations to the coefficient matrix of the saddle point problem than the HSS preconditioner, meanwhile, their algorithmic costs are comparable with the HSS preconditioner. The corresponding iteration methods of the two preconditioners are proved convergent unconditionally and the optimal iteration parameters are determined which result in fast convergence. Eigenvalue distributions of the preconditioned matrices are described and the bounds of the degree of their minimal polynomials are obtained. It is shown that the two variants have much better convergence properties and spectrum distributions than the HSS preconditioner.

The remainder of this paper is organized as follows. In Section 2, the descriptions of the two variants of the HSS preconditioner and the implementations of the corresponding iteration schemes are given. In Section 3, convergence analyses of the corresponding iteration methods are presented and the optimal iteration parameters are obtained. In Section 4, results concerning the spectral properties of the two preconditioned matrices and their minimum polynomials are established. In Section 5, numerical experiments from Stokes equations are presented to compare the two new preconditioners with the HSS preconditioner and some other preconditioners. Finally, we give some concluding remarks in Section 6.

2. Two variants of the HSS preconditioner and their iteration implementations

By simply switching positions of some sub-matrices of the HSS preconditioner (1.5), we will obtain the following two modified HSS (MHSS) preconditioners:

$$\mathscr{P}_{\text{MHSS-I}} = \frac{1}{2\alpha} \begin{pmatrix} \alpha I + A & 0 \\ 0 & 2\alpha I \end{pmatrix} \begin{pmatrix} \alpha I & B^T \\ -B & C \end{pmatrix} \tag{2.1}$$