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Generalized shift-splitting preconditioners for nonsingular and singular generalized saddle point problems

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ABSTRACT

Based on the shift-splitting technique, a class of generalized shift-splitting preconditioners are proposed for both nonsingular and singular generalized saddle point problems. The generalized shift-splitting preconditioner is induced by a generalized shift-splitting of the generalized saddle point matrix, resulting in a generalized shift-splitting fixed-point iteration. Theoretical analyses show that the generalized shift-splitting iteration method is convergent and semi-convergent unconditionally for solving the nonsingular and the singular generalized saddle point problems, respectively. Numerical experiments of a model Navier–Stokes problem are implemented to demonstrate the feasibility and effectiveness of the proposed preconditioners.

1. Introduction

In many areas of scientific computing and engineering applications, we need to solve the following large sparse generalized saddle point problems

$$\mathcal{A}x \equiv \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsymmetric positive definite (i.e., the symmetric part $H = \frac{1}{2}(A + A^T)$ is positive definite), $B \in \mathbb{R}^{m\times n}$ $(m \le n)$ is a rectangular matrix, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semi-definite, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are given vectors. For an overview of its applications, we refer to [1,2] and references therein.

Since the (1, 1) block matrix A is nonsingular, the generalized saddle point matrix $\mathcal{A}$ can be decomposed as

$$\mathcal{A} = \begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C + BA^{-1}B^T \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}. \tag{1.2}$$

It readily follows from the block decomposition (1.2) that the generalized saddle point matrix $\mathcal{A}$ is nonsingular if and only if $C + BA^{-1}B^T$ is. If B has full row rank, that is $rank(B) = m$, then the generalized saddle point matrix $\mathcal{A}$ is nonsingular. However, in many applications, the matrix B is rank deficient, that is $rank(B) < m$. For such case, $C + BA^{-1}B^T$ is invertible if and only if $null(C) \bigcap null(B^T) = \{0\}$. Therefore, if $null(C) \bigcap null(B^T) \ne \{0\}$, then the generalized saddle point matrix $\mathcal{A}$ is singular. Here, $null(\cdot)$ denotes the null space of the corresponding matrix.

A lot of efficient methods (including the null space method, the coupled direct solver, the stationary iterative method and the Krylov subspace method) have been proposed for solving the generalized saddle point problems (1.1); see also [2]

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for a general introduction to the different solution methods. For its property of large and sparsity, it may be attractive to use iterative methods. In the past decades, a lot of efficient stationary iterative methods have been proposed to solve (1.1), such as the Hermitian and skew-Hermitian splitting (HSS) iterative method [3,4], the deteriorated positive definite and skew-Hermitian splitting iterative method [5,6], the block alternating splitting implicit iteration method [7], the local HSS iterative method [8], the parameterized inexact Uzawa iterative method [9], the indefinite iterative method [10] and so on. Based on above mentioned stationary iterative methods, a lot of efficient preconditioners, including the block diagonal and block triangular preconditioners [11], the HSS-based preconditioners [4,12] and the constraint preconditioners [10,13–15], can be constructed for accelerating the convergence rate of the Krylov subspace methods. In this paper, based on the shift-splitting iteration methods studied in [16–19], we focus on a class of *generalized shift-splitting* (GSS) preconditioners

$$\mathcal{P}_{GSS} = \frac{1}{2}\begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I + C \end{bmatrix}, \tag{1.3}$$

which are induced by the following generalized shift-splitting iteration method

$$\frac{1}{2}\begin{bmatrix} \alpha I + A & B^T \\ -B & \beta I + C \end{bmatrix}\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \alpha I - A & -B^T \\ B & \beta I - C \end{bmatrix}\begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.4}$$

for solving the generalized saddle point problems (1.1), where α and β are two positive real parameters, and I is the identity matrix with appropriate dimension. The idea of the shift-splitting iteration method was firstly proposed by Bai et al. [16] for solving a class of non-Hermitian positive definite systems of linear equations. Then it was extended by Cao et al. [17–19] to solve nonsingular saddle point problems, where A is positive definite, B is of full row rank and $C = 0$. For the nonsingular generalized saddle point problems (1.1) where A is positive definite, B is of full row rank and C is symmetric positive semi-definite, Salkuyeh et al. [20,21] proved that the GSS iteration method is convergent unconditionally. For a class of singular saddle point problems, where A is symmetric positive definite, B is rank deficient and $C = 0$, Chen et al. [22] proved that the GSS iteration method is semi-convergent to a solution if the singular linear system is consistent. For singular nonsymmetric saddle point problems where A is nonsymmetric positive definite, B is rank deficient and $C = 0$, the semi-convergence of the GSS iteration method was proved by Cao and Miao in [23], where a unified analysis was also presented for solving nonsingular nonsymmetric saddle point problems. However, there is no discussion on the GSS iteration method and the corresponding GSS preconditioner for the singular generalized saddle point problems where A is nonsymmetric positive definite and $C \neq 0$.

In this paper, the GSS preconditioner (1.3) is further studied for both the singular and the nonsingular generalized saddle point problems (1.1). It will be shown that the GSS iteration method (1.4) is convergent unconditionally for solving the nonsingular generalized saddle point problems and semi-convergent unconditionally for solving the singular generalized saddle point problems. Based on the convergence analysis of the GSS iteration method, we obtain that the GSS preconditioned matrix $\mathcal{P}_{GSS}^{-1}\mathcal{A}$ has a clustered eigenvalue distribution, which is a desirable property for Krylov subspace acceleration. It should be noted that our results are of considerable generality, including as particular cases almost all the results given in recent literatures.

The remainder of this paper is organized as follows. In Sections 2 and 3, the convergence and the semi-convergence of the GSS iteration method for solving the nonsingular and singular generalized saddle point problems are studied, respectively. Implementation aspects of the GSS preconditioner are also discussed in Section 2. Numerical experiments of a model Navier–Stokes problem are illustrated to show the feasibility and effectiveness of the proposed preconditioners in Section 4. Finally, we end this paper with a few concluding remarks in Section 5.

2. Convergence and preconditioning properties for nonsingular case

In this section, we study the unconditional convergence of the GSS iteration method for solving the nonsingular generalized saddle point problems (1.1), where A is nonsymmetric positive definite, B is of full row rank and C is symmetric positive semi-definite. Then the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_{GSS}^{-1}\mathcal{A}$ is obtained.

Let

$$\Omega = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \quad \text{and} \quad \mathcal{Q}_{GSS} = \frac{1}{2}(\Omega - \mathcal{A}) = \frac{1}{2}\begin{bmatrix} \alpha I - A & -B^T \\ B & \beta I - C \end{bmatrix},$$

then the GSS iteration method (1.4) is based on the following matrix splitting

$$\mathcal{A} = \mathcal{P}_{GSS} - \mathcal{Q}_{GSS}$$

and can be equivalently written as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \Gamma \begin{bmatrix} x_k \\ y_k \end{bmatrix} + c, \tag{2.1}$$