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On weak–strong uniqueness of solutions to the generalized incompressible Navier–Stokes equations ☆

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ABSTRACT

In the recent paper Li and Zhai (2010) proved the well-posedness of the Cauchy problem to the n-dimensional generalized incompressible Navier–Stokes equations with initial data u_0 belonging to the so-called Q-space $Q_{\alpha;\infty}^{\beta,-1}(\mathbb{R}^n)$ with $\beta \in (\frac{1}{2}, 1]$ and $\alpha \in [0, \beta)$. In this paper, by using the Littlewood–Paley theory, we prove the weak–strong uniqueness between weak solution and Li–Zhai's strong solution for the n-dimensional generalized incompressible Navier–Stokes equations.

1. Introduction

In this paper, we are interested in the Cauchy problem of n-dimensional generalized incompressible Navier–Stokes (GNS) equations in the half-space $\mathbb{R}^n \times (0, \infty)$ (with $n \geq 3$) without external force and with initial data u_0 satisfying $\operatorname{div} u_0 = 0$:

$$\begin{cases} u_t + (-\Delta)^\beta u + (u \cdot \nabla)u + \nabla P = 0 \\ \operatorname{div} u = 0. \end{cases} \tag{1.1}$$

Here $\beta \in (\frac{1}{2}, 1]$. $u = u(x, t) = (u^1(x, t), \ldots, u^n(x, t))$ and $P = P(x, t)$ stand for, respectively, the fluid velocity and the pressure. The fractional Laplace operator $(-\Delta)^\beta$ with respect to space variable x is a Riesz potential operator defined as usual through Fourier transform as $\mathcal{F}((-\Delta)^\beta f)(\xi) = |\xi|^{2\beta}\mathcal{F}f(\xi)$, where $\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)\mathrm{d}x$.

System (1.1) is a generalization of the usual incompressible Navier–Stokes (NS) equations by replacing the Laplace operator $-\Delta$ in the NS equations by a general fractional Laplace operator $(-\Delta)^\beta$ (see Wu [1–3]). In the case when $\beta = 1$, for $u_0 \in L^2(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$, the global existence of weak solution was established by the papers of Leray [4] and Hopf [5], but it is still unknown whether the Leray–Hopf's weak solution to the NS equations is unique, and there are a great deal of papers devoted to the study of uniqueness of weak solutions. It is proved that if u and v are two Leray–Hopf's weak solutions with the same initial data and one of the solutions (say u) satisfies

$$u \in L^q(0, T; L^p(\mathbb{R}^n)) \quad \text{with } \frac{2}{q} + \frac{n}{p} \leq 1,\ n \leq p \leq \infty, \tag{1.2}$$

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then $u \equiv v$ on $[0, T]$, we refer to [6–8] and the references therein. In 1964, another class of strong solutions to the NS equations was constructed by the pioneered papers of Fujita and Kato [9,10], and their works inspired extensive study in the following years on the well-posedness of the NS equations in various critical spaces, see Kato [11], Cannone [12], Koch and Tataru [13], Lemarié-Rieussetc [14], Xiao [15,16] and so on. Particularly, Cannone constructed the solution for the NS equations with the initial data $u_0 \in \dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ for $n < p < \infty$ (here $\dot{B}_{p,q}^{s}(\mathbb{R}^n)$ is the homogeneous Besov space, please see the definition in the next section); Koch and Tataru established the well-posedness for the initial data in $BMO^{-1}(\mathbb{R}^n)$. Recently, Xiao in [15,16] generalized the results of Koch and Tataru to a new so-called Q-space $Q_{\alpha,\infty}^{-1}(\mathbb{R}^n)$ for $\alpha \in [0, 1)$. It is natural to question that a strong solution and a Leray–Hopf's weak solution unique among all the possible Leray–Hopf's weak solutions are associated to the same initial data. Let $\mathscr{H} := \cup_{\gamma>0} H^{\gamma}(\mathbb{R}^n)$, $\mathscr{B} := \cup_{n<p<\infty} \dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^n)$ and $\mathscr{B}(T)$ be the space of tempered distributions f on $[0, T) \times \mathbb{R}^n$ with the norm

$$\|f\|_{\mathscr{B}(T)} := \sup_{j\in\mathbb{Z}} \left\{ 2^{j\left(-1+\frac{n}{p}\right)} \|\Delta_j f\|_{L^{\infty}(0,T;L^p)} + 2^{j\left(1+\frac{n}{p}\right)} \|\Delta_j f\|_{L^1(0,T;L^p)} \right\}.$$

Chemin [17] established that for the initial data $u_0 \in \mathscr{H} \cap \mathscr{B}$, the strong solution $u \in \mathscr{B}(T)$ of the NS equations is in fact unique among the Leray–Hopf's weak solution associated with u_0, thus to say, if v is a Leray–Hopf's weak solution with the initial data u_0, then $u = v$ on $[0, T]$. Essentially, Chemin proved the weak–strong uniqueness for Cannone's solution. Very recently, Dong and Zhang [18] studied the weak–strong uniqueness for Koch–Tataru's solution, and obtain that for $u_0 \in \mathscr{H} \cap \widetilde{BMO}^{-1}$, the Koch–Tataru's solution u is unique among the Leray–Hopf's weak solutions associated with u_0. Here, the space $\widetilde{BMO}^{-1}$ is the closure of smooth, compactly supported functions with respect to the norm $\|\cdot\|_{BMO^{-1}}$. For more results on the issue of the weak–strong uniqueness of solutions to the NS equations, we refer to Dubios [19], Germain [20], Lemarié-Rieusset [14] and the references therein.

For the general case, the fractional Laplacian $(-\Delta)^{\beta}$ can in principle arise from modeling real physical phenomena, and for more details on it can be found in Chapter 5 of [21] and papers [22,1,2]. One important advantage of working with the GNS equations is that it allows simultaneous consideration of its solutions corresponding to a range of β's. For instance, the 3D GNS equations with any $\beta > \frac{5}{4}$ always poses global classical solutions [22]. Wu [1] proved that when $\beta \geq \frac{1}{2} + \frac{n}{4}$ with $n \geq 3$, the GNS equations (1.1) still possess unique global classical solutions. In [1], Wu also proved the existence of global-in-time weak solutions. For the existence of strong solutions, Wu [2,3] established the local existence and uniqueness results of (1.1) in Besov spaces for the case of $\beta < \frac{1}{2} + \frac{n}{4}$. Yu and Zhai [23] studied the well-posedness of (1.1) in the largest critical Besov spaces $\dot{B}_{\infty,\infty}^{-(2\beta-1)}(\mathbb{R}^n)$ with $\beta \in (\frac{1}{2}, 1)$. Liu, Zhao and Cui [24] obtained the global existence and stability of strong solutions for system (1.1) with small initial data u_0 belonging to the critical pseudomeasure space $PM^a(\mathbb{R}^n)$ (with $a \geq n - (2\beta - 1)$ a given parameter, and $\frac{1}{2} < \beta < \frac{n+2}{4}$), where $PM^a(\mathbb{R}^n)$ is defined by

$$PM^a(\mathbb{R}^n) := \left\{ f \in \mathscr{S}' : \widehat{f} \in L^1_{\text{loc}}(\mathbb{R}^n),\ \|f\|_{PM^a} := \operatorname{ess\,sup}_{\xi\in\mathbb{R}^n} |\xi|^a |\widehat{f}(\xi)| < \infty \right\}.$$

In recent paper [25], inspired by Koch and Tataru [13] and Xiao [15], Li and Zhai proved the well-posedness of the GNS equations (1.1) with initial data in the new critical space $Q_{\alpha,\infty}^{\beta,-1}(\mathbb{R}^n)$ with $\beta \in (\frac{1}{2}, 1]$ and $\alpha \in [0, \beta)$, and they also proved that the strong solution is spatial smooth. The regularity of strong solutions to the GNS equations (1.1) was studied by Dong and Li [26]. Similar to the studies of the NS equations, the weak–strong uniqueness of solutions to the GNS equations is still studied by many researchers. In Constantin and Wu [27], condition similar to (1.2) for the uniqueness was obtained, more precisely, for $\beta > \frac{1}{2}$ and $u_0 \in L^2(\mathbb{R}^n)$, if the solution u of the GNS equations satisfies

$$u \in L^q(0, T; L^p(\mathbb{R}^n)) \quad \text{with } \frac{2\beta}{q} + \frac{n}{p} \leq 2\beta - 1,\ \frac{n}{2\beta - 1} < p \leq \infty,$$

then u is unique among the weak solutions associated with u_0. In paper [28], Wu and Fan generalized Constantin and Wu's results, and proved that for $\beta > \frac{1}{2}$ and $u_0 \in L^2(\mathbb{R}^n) \cap \dot{B}_{p,q}^{\frac{n}{p}-2\beta+1}(\mathbb{R}^n)$ with $\frac{n}{p} + \frac{2\beta}{q} \geq \beta$, assume that the strong solutions u satisfy

$$u \in L^q\left(0, T; \dot{B}_{p,q}^{\frac{n}{p}+\frac{2\beta}{q}-2\beta+1}(\mathbb{R}^n)\right) \quad \text{with } \frac{n}{p} + \frac{2\beta}{q} > \beta,\ 2 \leq p < \infty,\ 2 < q < \infty,$$

then u is unique among the weak solutions associated with u_0.

Motivated by the papers of Chemin [17], Dubois [19], Germain [20] and Dong and Zhang [18] on the NS equations, and Constantin and Wu [27], Wu and Fan [28] on the GNS equations, the purpose of this paper is to study the weak–strong uniqueness of solutions to the GNS equations (1.1). We hope that our study of the GNS equations (1.1) will broaden our view on the issue of weak–strong uniqueness for the NS equations. Before stating our main results, let us first recall the definition of the weak solution of the GNS equations (1.1) according to Wu's definition [1].

Definition 1.1. We say u is a weak solution of the generalized Navier–Stokes equation (1.1) if u satisfies the following properties: