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The optimal error estimate and superconvergence of the local discontinuous Galerkin methods for one-dimensional linear fifth order time dependent equations

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ABSTRACT

In this paper, we investigate the optimal error estimate and the superconvergence of linear fifth order time dependent equations. We prove that the local discontinuous Galerkin (LDG) solution is $(k+1)$th order convergent when the piecewise P^k space is used. Also, the numerical solution is $(k+\frac{3}{2})$th order superconvergent to a particular projection of the exact solution. The numerical experiences indicate that the order of the superconvergence is $(k+2)$, which implies the result obtained in this paper is suboptimal.

1. Introduction

In this paper, we consider linear fifth order time dependent equation given by

$$\begin{aligned} &u_t + \alpha u_x + \beta u_{xxxxx} = 0, \\ &u(x,0) = u_0(x) \end{aligned} \tag{1.1}$$

with the periodic boundary condition, where α, β are constants. We study the superconvergence of the local discontinuous Galerkin solution towards a particular projection of the exact solution.

The fifth order nonlinear KdV equation

$$u_t + uu_x + \gamma u_{xxx} + \delta u_{xxxxx} = 0, \tag{1.2}$$

which is a model for weakly nonlinear waves in a wide variety of media, for instance, long waves in shallow liquid under ice cover [1]. Since the LDG method of (1.2) has two more auxiliary variables than those of Eq. (1.1), which makes the proof of superconvergence much more complicated. We only consider Eq. (1.1), which is the linearized model of (1.2) when $\gamma = 0$.

The discontinuous Galerkin (DG) methods belong to a class of finite element methods using the piecewise polynomial spaces for both the numerical solutions and the test functions. Since the piecewise polynomials used in the methods permit complete discontinuity across the element interface, the DG methods easily accommodate arbitrary h–p adaptivity. These methods also allow arbitrarily unstructured meshes and have a compact stencil. The DG method was first used to solve neutron equations [2]. Then, Cockburn et al. developed the Runge–Kutta discontinuous Galerkin (RKDG) methods for solving hyperbolic conservation laws in a series of papers [3–6]. The local discontinuous Galerkin (LDG) method is an extension of the DG method aiming at solving the equations with high order spatial derivatives. The first LDG method was constructed

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by Cockburn and Shu to solve convection–diffusion equations [7]. It was motivated by the successful numerical experiments of Bassi and Rebay for the compressible Navier–Stokes equations [8]. Then the LDG methods were applied to solve various partial differential equations with higher order derivatives including nonlinear one-dimensional and two-dimensional KdV type equations [9,10], the general fifth-order KdV type equations [11] and Ito-type coupled KdV equations [12]. See review paper [13] for more details. The LDG methods for KdV-type equations was first developed in [9], in which the sub-optimal L^2 error estimates were obtained. In [14], the optimal L^2 error estimates were derived by Xu and Shu by handling the jump terms at the cell boundaries which appear because of the discontinuous nature of the finite element space for the LDG methods.

In recent years, there are a lot of efforts to obtain the superconvergence results for the DG methods and the LDG methods. In [15,16], Adjerid et al. explored the superconvergence of the DG method for the ordinary differential equations and the steady hyperbolic problems at the Radau points. Based on Fourier analysis, Cheng and Shu proved that the DG solution is $(k+\frac{3}{2})$th order superconvergent to the particular projection of the exact solution when the linear polynomials on the uniform meshes are used for linear conservation laws with the periodic condition [17]. The results were extended by Cheng and Shu in [18] to the general polynomial degree k, non-uniform regular meshes, for the technique used in [18] is a finite type. The proof of the superconvergence of the LDG solution towards a particular projection of the exact solution for convection–diffusion equations was also given in [18]. Then the same technique was used to obtain the superconvergence of LDG method for linearized Korteweg–de Vries equations [19] and a class of fourth-order equations [20]. But the numerical results show that the convergence orders obtained in [18–20] are not optimal. In [21,22], Yang and Shu demonstrate that the orders of the superconvergence in [18] increase half-order for both linear conservation laws and linear parabolic equations by considering the dual problem of the original problem and constructing particular initial projection. They also prove that the DG and LDG solutions converge at a rate of $(k+2)$ at Radau points. In [23,24], Cao and Zhang carry out a different framework for proving superconvergence at Radau points. Further more, they obtain a $(k+1)$th superconvergence rate for the derivative approximation. Later Cao and Zhang extend their results to two-dimensional hyperbolic equations in [25]. For the nonlinear problems, Meng and Shu proved that the error between the DG solution and a particular projection is $(k+\frac{3}{2})$th superconvergent for the scalar nonlinear conservation laws, when the upwind fluxes are used [26].

In this paper, we study the optimal error estimate and the superconvergence of linear fifth order time dependent equations. In [18–20], two functionals were introduced to estimate the superconvergence which made the proof a little involved. Thanks to the lemma in [21], we give a simpler proof. Of course the superconvergence order is $(k+\frac{3}{2})$. And the numerical experiments indicate that our result is suboptimal. This paper is organized as follows: in Section 2, we introduce the LDG method for linear fifth order equation and some notations; in Section 3, we achieve the optimal error estimate of the LDG method for Eq. (1.1); in Section 4, we give the proof of the superconvergence of the LDG solution towards the particular projection of exact solution; in Section 5, some numerical results are provided; the conclusion and the future works are given in Section 6; In the Appendix, we give the proof of a lemma.

2. Local discontinuous Galerkin method

2.1. LDG method for fifth order equations

In this subsection, we will present the LDG method for the Eq. (1.1). First we divide the interval $I=[0,2\pi]$ into N subintervals as follows:

$$0=x_{\frac{1}{2}}<x_{\frac{3}{2}}<\cdots<x_{N+\frac{1}{2}}=2\pi.$$

Then we denote each subinterval by $I_j=(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}})$ and the centre of the subinterval $x_j=\frac{1}{2}(x_{j-\frac{1}{2}}+x_{j+\frac{1}{2}})$. We also set $h_j=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$. The left and right limits of the function v_h at the discontinuity point $x_{j+\frac{1}{2}}$ are denoted by $(v_h)^-_{j+\frac{1}{2}}$ and $(v_h)^+_{j+\frac{1}{2}}$, respectively. And we denote the jump of the function $v_h(x)$ at the element interface as $[v_h]_{j+\frac{1}{2}}=(v_h)^+_{j+\frac{1}{2}}-(v_h)^-_{j+\frac{1}{2}}$. Let $h=\max_j h_j$ and $h_{\min}=\min_j h_j$. We assume that our mesh is regular, which means that there exists a constant λ such that $\lambda=h/h_{\min}$. Clearly, the mesh is uniform, when $\lambda=1$.

We denote the finite element space by

$$V_h^k=\{v:v|_{I_j}\in P^k(I_j)\},$$

where $P^k(I_j)$ is the space of polynomials of degree at most k on I_j.

To construct the LDG method, we need four auxiliary variables s,r,p,q, and rewrite Eq. (1.1) as a first order linear system

$$u_t+(\alpha u+\beta s)_x=0,\qquad s=r_x,\qquad r=p_x,\qquad p=q_x,\qquad q=u_x. \tag{2.1}$$

Then we need to find $u_h,q_h,p_h,r_h,s_h\in V_h^k$ such that for any $\rho,\zeta,\eta,\psi,\phi\in V_h^k$

$$((u_h)_t,\rho)_j+\alpha(\hat{u}_h\rho^-|_{j+\frac{1}{2}}-\hat{u}_h\rho^+|_{j-\frac{1}{2}}-(u_h,\rho_x)_j)+\beta(\tilde{s}_h\rho^-|_{j+\frac{1}{2}}-\tilde{s}_h\rho^+|_{j-\frac{1}{2}}-(s_h,\rho_x)_j)=0, \tag{2.2a}$$

$$(s_h,\zeta)_j-\tilde{r}_h\zeta^-|_{j+\frac{1}{2}}+\tilde{r}_h\zeta^+|_{j-\frac{1}{2}}+(r_h,\zeta_x)_j=0, \tag{2.2b}$$