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Numerical approach to monotone variational inequalities by a one-step projected reflected gradient method with line-search procedure

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ABSTRACT

In this paper we revisit a recent approach to classical monotone variational inequalities by means of a projected reflected gradient-type method in $\mathbb{R}^N$. A line-search procedure is incorporated for possible varying step-sizes and no Lipschitz-continuity condition on the operator is required. A main feature of the proposed method is that it requires only one projection step onto the feasible set at each iteration.

1. Introduction

Throughout this paper, C is a closed convex set of $\mathbb{R}^N$, while $\langle .,. \rangle$ and $|\cdot|$ denote the usual inner product of $\mathbb{R}^N$ and its induced norm, respectively. We aim at revisiting the computation of an element of the solution set, denoted by S, of the classical variational inequality problem:

$$\text{find } u \in C \text{ such that } \langle F(u), v-u \rangle \geq 0 \quad \forall v \in C, \tag{1.1}$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is a given mapping such that:

$$S \neq \emptyset; \tag{1.2a}$$

$$F \text{ is monotone over } \mathbb{R}^N\text{, i.e., for all } x, y \in \mathbb{R}^N\text{: } \langle F(x) - F(y), x - y \rangle \geq 0; \tag{1.2b}$$

$$F \text{ is continuous over } \mathbb{R}^N. \tag{1.2c}$$

This formalism (first introduced by Stampacchia in [1]) is well-known to provide a unified framework for the study of many significant real-world problems arising in mechanics, economics and so on (see, e.g., [2–5] and the references therein). It is worthwhile underlining that (1.1) gave rise to many algorithmic solutions depending on the structure of C and the properties of F. In particular when F and C do not possess any special structure the proposed numerical approaches to (1.1) are mainly based upon projection techniques onto C; see, e.g., [6].

Definition 1.1. The metric projection $P_C : \mathbb{R}^N \to C$ is the operator defined for all $x \in \mathbb{R}^N$ by $P_C x := \text{argmin}_{z \in C} |z - x|$.

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The use of such techniques was suggested by the following equivalent fixed point formulation of (1.1): find $u \in C$ such that $u = P_C(u - \lambda F(u))$, where λ is any positive real number.

Let us recall that the oldest method for solving (1.1) in a general framework is the following so-called extra-gradient method (introduced by Korpelevich [7]):

$$\bar{x}_n = P_C(x_n - \lambda_n F(x_n)), \qquad x_{n+1} = P_C(x_n - \lambda_n F(\bar{x}_n)), \tag{1.3}$$

where the step-sizes (λ_n) are positive real numbers. This algorithm involves two projection steps per iteration and its convergence was initially established under monotonicity and Lipschitz continuity of F for constant step-sizes (depending on the Lipschitz constant of F). Later on, the extra-gradient method has been enhanced through several extensions involving Armijo-type rules (see, e.g., Khobotov [8], Marcotte [9], Sun [10], Iusem [11], Tseng [12]) and outer approximation techniques (see, e.g., Solodov and Svaiter [13]). These latter strategies are specifically well-adapted to the situation when F is not Lipschitz continuous or no estimate of the Lipschitz constant is available.

This paper is mostly concerned with the case when the evaluation of the projection operator onto C is computationally expensive. A crucial feature regarding the design of numerical methods related to such a context is to minimize the number of evaluation of P_C per iteration. As famous algorithms with such interesting features we mention that ones discussed by Iusem–Svaiter [14] and Solodov–Svaiter [13]. These methods (that involve a more effective line-search procedure) were able to drop the Lipschitz continuity condition even for a pseudo-monotone mapping F (also see Iusem–Pérez [15] for extension to nonsmooth cases of F). These latter algorithms involve one projection onto C at each iteration together with another projection onto either C or onto its intersection with some hyperplane, but, contrary to what was done so far, no additional projection is needed in the line-search procedure. Modified extra-gradient methods with only one projection onto C per iteration has been also investigated. As an example, a special case (relative to (1.1)) of the general method proposed by Tseng [16] formally involves one projection step but its convergence was established by using an Armijo–Goldstein-type stepsize rule for which the trial values of step-sizes require additional evaluations of P_C. Other examples of such methods combine one projection onto C per iteration together with a cheaper projection step onto some hyperplane (see, e.g., Censor, Gibali and Reich [17], Malitsky and Semenov [18]). However, their convergence was stated under the condition of Lipschitz continuity of F for step-size rules that depend on the Lipschitz constant.

In this work we focus our attention on an alternative approach to (1.1) based on the following so-called projected reflected gradient method recently proposed by Malitsky [19]:

$$y_n = 2x_n - x_{n-1}, \qquad x_{n+1} = P_C(x_n - \lambda_n F(y_n)), \tag{1.4}$$

with positive step-sizes (λ_n). This latter algorithm (formally) involves only one evaluation for each of the operators P_C and F per iteration. It was investigated in the context of a Hilbert space together with conditions of monotonicity and L-Lipschitz continuity of F (for some positive value L) over the whole space. The convergence of (1.4) was obtained is the special case of a constant step-size $\lambda \in \left(0, \frac{\sqrt{2}-1}{L}\right)$. A convergence result was also established for varying step-sizes (λ_n) given by a specific procedure that (unfortunately) involves the computation of additional projections onto C. Nonetheless numerical experiments have been performed in [19] showing interesting and promising features (regarding the computational viewpoint) comparing with other numerical strategies.

The following variant of (1.4) has been also investigated (in a Hilbert space under the same conditions on F) in [20]:

$$y_n = x_n + \theta_n(x_n - x_{n-1}), \qquad x_{n+1} = P_C(x_n - \lambda_n F(y_n)), \tag{1.5}$$

where $\theta_n = \frac{\lambda_n}{\delta \lambda_{n-1}}$ for some $\delta \in (0, 1]$. The convergence of (1.5) was stated under the general conditions below on the step-sizes:

$$\lambda_n |F(y_n) - F(y_{n-1})| \le \epsilon\delta(\sqrt{2} - 1)|y_n - y_{n-1}|, \quad \text{for some } \epsilon \in (0, 1), \tag{1.6a}$$

$$\lambda_n \le \lambda_{n-1}\left(\delta + \frac{\lambda_{n-1}}{\lambda_{n-2}}\right)^{1/2}, \tag{1.6b}$$

$$(\lambda_n) \subset [\bar{\mu}, \bar{\nu}] \quad \text{for some positive } \bar{\mu} \text{ and } \bar{\nu}. \tag{1.6c}$$

Our purpose here is to propose an enhanced variant of (1.5) for solving (1.1) under condition (1.2) (so we drop the Lipschitz continuity of the operator F) by incorporating a linesearch procedure that does not require any additional evaluation of P_C.

2. The algorithm and its convergence

For the sake of simplicity, we sometimes use the following notations: $\dot{x}_n = x_n - x_{n-1}$ and $\dot{y}_n = y_n - y_{n-1}$. In order to compute a solution of (1.1) we focus on the sequence (x_n) generated by the following method.