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On ground state solutions for a Kirchhoff type equation with critical growth ☆

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ABSTRACT

In this paper, with the aid of variational method and concentration-compactness principle, a positive ground state solution is obtained for a class of Kirchhoff type equations with critical growth

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u = u^5+\lambda k(x)u^{q-1}, \ x\in\mathbb{R}^3, \\ u\in D^{1,2}(\mathbb{R}^3), \end{cases}$$

where $\lambda > 0$ is a parameter, $2 < q < 6$ and k satisfies some conditions.

1. Introduction and main results

We are concerned with the existence of positive solutions for the following Kirchhoff type problems involving the critical growth

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u = u^5+\lambda k(x)u^{q-1}, \quad x\in\mathbb{R}^3, \\ u\in D^{1,2}(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a, b > 0$ and $2 < q < 6$, $\lambda > 0$ is a real number.

It is known that Kirchhoff type equations have a strong physical meaning because they appear in mechanics models. For example, for problem

$$\begin{cases} -\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

is related to the stationary analogue of the Kirchhoff equation which was proposed by Kirchhoff in [1] as a generation of the d'Alembert's equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|dx\right)\frac{\partial^2 u}{\partial x^2}=0$$

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for free vibrations of elastic strings. In [2], Lions introduced an abstract functional analysis framework to the equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2\right)\Delta u = f(x,u).$$

After that there are abundant results about it, for example, in [3], Perera and Zhang obtained nontrivial solutions via the Yang index and critical group; Zhang and Perera in [4] obtained multiple and sign-changing solutions by using the invariant sets of descent flow. In particular, when the nonlinearity $f(x,u)$ is critical growth in Eq. (1.2), some important and interesting results can be found when $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain (see [5–15] and their references therein).

Problem (1.2) on unbounded domains involving critical growth has attracted a lot of attention, for example, in [16–20], the existence of positive solutions of (1.2) has been obtained via the variational methods.

In recent years, there have been many papers concerned with the existence of positive ground state solutions for Kirchhoff type problems. Results relating to these problems can be found in [15,19,21,22].

Wang et al. in [19] have considered the following Kirchhoff type problem involving critical growth:

$$\begin{cases} -\left(\varepsilon^2 a + b\varepsilon\displaystyle\int_{\Omega} |\nabla u|^2 dx\right)\Delta u + M(x)u = |u|^4 u + \lambda f(u), & \text{in } \mathbb{R}^3,\\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $f \geq 0$ satisfies the following conditions:

(A_1) $f \in C^1(\mathbb{R}), f(t) = o(t^3)$ as $t \to 0$;
(A_2) $\frac{f(t)}{t^3}$ is strictly increasing on interval $(0,\infty)$;
(A_3) $|f(t)| \leq c(t + |t|^{p-1})$ for some $c > 0, 4 < p < 6$.

Under the assumptions of the above, they showed the existence of positive ground state solutions by the variational method.

Very recently, in [22], Li and Ye have proved the existence of $2 < p < 5$ such that the problem

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = |u|^{p-1}u, & x \in \mathbb{R}^3,\\ u \in H^1(\mathbb{R}^3) \end{cases}$$

has a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma.

The main purpose of this paper is to establish the existence of a positive ground state solution of (1.1) by means of the variational method and concentration-compactness principle.

Before stating our results, we need the following assumptions:

(k_1) $k \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ and $k(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $k \not\equiv 0$;
(k_2) There exist $x_0 \in \mathbb{R}^3$ and $\delta, \rho_1 > 0$ such that $k(x) \geq \delta|x - x_0|^{-\beta}$ for $|x - x_0| < \rho_1$ and $0 < \beta < 3$.

Now our main results can be described as follows:

Theorem 1.1. *Assume the hypotheses (k_1)–(k_2) and $2 < q < 4$ with $3 - \frac{2q}{3} < \beta < 3$ hold, then there exists $\lambda_* > 0$ such that problem (1.1) has at least one positive ground state solution for all $0 < \lambda < \lambda_*$.*

Theorem 1.2. *Assume the hypotheses (k_1)–(k_2) and $4 \leq q < 6$ with $0 < \beta < 3$ hold, then problem (1.1) has at least one positive ground state solution for all $\lambda > 0$.*

Throughout this paper, we make use of the following notations:

- The norm in $D^{1,2}(\mathbb{R}^3)$ equipped with the norm $\|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$, we shall simply write $|\cdot|_p$ as the norm in $L^p(\mathbb{R}^3)$.
- $\to$ (respectively, $\rightharpoonup$) denotes strong (respectively, weak) convergence;
- $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line;
- We denote by S_r (respectively, B_r) the sphere (respectively, the closed ball) of center zero and radius r, i.e., $S_r = \{u \in D^{1,2}(\mathbb{R}^3) : \|u\| = r\}$, $B_r = \{u \in D^{1,2}(\mathbb{R}^3) : \|u\| \leq r\}$;
- Denoting by S the best constant of the above inequality, namely

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{\frac{1}{3}}}. \tag{1.3}$$

This work is organized as follows. In the next section we present some preliminary results. In Section 3, we give the proof of Theorems 1.1–1.2.