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High-order Discontinuous Galerkin Methods for a class of transport equations with structured populations

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ABSTRACT

In this paper we analyze a discontinuous Galerkin finite element method for approximating solutions to transport equations with certain nonlinearities. We consider models for age-structured populations allowing for a nonlinear removal rate with non-local boundary conditions on the in-flow boundary. The method employs a stabilizing term over the interior edges allowing for convergence in a stronger than usual norm. We establish convergence rates for general higher order basis functions and provide numerical examples consistent with this result.

1. Introduction

An interesting class of transport equations involves those in which the inflow boundary condition depends on the amount of the substance present within the domain of advection. As an example, consider a channel in which a valve at the inflow-end turns on or off depending on the level of water in the channel.

Such models arise frequently in biology and ecology in the study of population dynamics where the species are structured in some manner (typically by age). In this setting, the models fall under the broad class of the Gurtin–MacCamy or Lotka–McKendrick models and have been studied extensively [1–4]. Areas where such models arise include cell physiology [5], ecology, e.g. host-parasitization dynamics [6] and epidemiology [7].

Let $u(a, t)$ represent a concentration density as a function of the independent variables (a, t). The system we consider is

$$\alpha_1 u_t + \alpha_2 u_a + \lambda(a, t, u) = f(a, t), \quad t > 0,\ 0 < a < \hat{A} \tag{1a}$$

$$B(t) = u(0, t) = \int_0^{\hat{A}} \beta(a) u(a, t)\, da, \quad t > 0 \tag{1b}$$

$$u(a, 0) = u_o(a), \quad 0 < a < \hat{A}. \tag{1c}$$

The variable $\hat{A}$ may be finite or infinite and β is a bounded non-negative real-valued function. The coefficients α_1, α_2 are positive functions of the independent variables. In most papers, these coefficients are set to 1 and represent the characteristic directions for (1a); in the present work we allow for more general characteristics, and denote $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. With this

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notation, if α_1, α_+2 are constants, $u_{\boldsymbol{\alpha}} = \nabla \cdot (\boldsymbol{\alpha} u) = \alpha_1 u_t + \alpha_2 u_a$ and (1a) becomes

$$u_{\boldsymbol{\alpha}} + \lambda(a, t, u) = f(a, t).$$

The analysis of the numerical method in this paper is valid even for non-constant α_i.

In the aforementioned age-structured population models, $u(a, t)$ denotes the age density at time t of individuals of age a. A biological interpretation of the source term $f(a, t)$ is an effect on the removal rate of the population which is due to external effects, and in which members of the species are equally affected regardless of their age. Another interpretation is in terms of a harvesting function [8]. The removal rate $\lambda(a, t, u) \geq 0$ is interpreted as removal due to natural death. The terminal age of individuals in the population is $\hat{A}$, and can be finite (e.g. for differentiated blood cells) or infinite (for stem cells). $B(t)$ is the birth rate of individuals. In the rest of this paper, we will adopt this biologically-inspired terminology for these variables.

In the context of age-structured population models [1,9], it is typical to consider model (1) with $f(a, t) \equiv 0$. In many prior works, a specific form of the removal rate $\lambda(t, a, u)$ is considered to be

$$\lambda(t, a, u) = \mu(a)u(a, t)$$

where the age-dependent mortality rate $\mu(a) > 0$ depends only on a. In other words, the removal rate is linear in the unknown u. In the present paper we allow for a more general form of removal,

$$\lambda(t, a, u) = \mu(a)q(u(a, t))$$

where $q(z)$ may be nonlinear in its argument. Consequently the removal rate may become nonlinear in u.

Finally, if the terminal age $\hat{A}$ is finite, then in age-structured applications λ is allowed to become unbounded as $a \to \hat{A}$ [10]. There are also investigations of models like (1) with bounded mortality, e.g. [11]. If $\hat{A}$ is infinite, this rate must remain positive and growing. In this paper we allow the mortality rate to be unbounded at terminal age.

The well-posedness properties of such models have been extensively studied under various assumptions on the removal rate $\lambda(a, t, u)$ and birth rate β [1,4,12,13]. With these assumptions, a range of numerical approximation strategies have been suggested for (1), including collocation and finite difference approaches [14–20]. In addition, semi-discrete models (discrete in age, continuous in time) have been studied in [9,21]. A space–time continuous finite element strategy is investigated in [22]. Another popular strategy is to integrate the system in the age variable, leading to a delay differential equation; this is then discretized [5]. In [9], a continuous-in-time, DG-in-age method is analyzed and used for (1).

The analysis in this paper was inspired by prior work on DG methods for linear hyperbolic problems, including those in [23–28]. Since we allow the removal rate $\lambda(a, t, u)$ to be nonlinear in the density u, we cannot directly use the ideas for the analysis of finite element algorithms for linear hyperbolic PDEs.

The use of discontinuous Galerkin methods for age-structured populations models has been discussed by other authors. In this paper, we extend the ideas found in [10,23,24]. The authors in [24] consider an hp-DG method for the simplified model where $f(t, a) \equiv 0$ and $\lambda(t, a, u)$ is linear in $u(t, a)$. Similarly, in [10], model (1) is considered with $f(t, a) \equiv 0$, the direction of the characteristics is $(1, 1)$, and a specific form of $\lambda(t, a, u)$, namely, $\lambda(t, a, u) = \mu(a)u(a, t)$. The focus is on an analysis of an explicit method which allows for a block-wise solve, and which moreover guarantees a non-negative solution. This is achieved by enforcing a CFL-like condition on the mesh. In our work, we consider the more general version of the problem (1), allowing for forcing terms and nonlinear behavior in the removal rate. Similar to [10,24], our proposed stabilized method is also in the high-order setting. Unlike in [10], our scheme does not remove non-locality in an explicit way. Our formulation allows for characteristics in directions other than $(1, 1)$; while age-structured populations provide an important area of application for the models presented in (1) (where the $(1, 1)$ direction is natural), our goal is to provide a simple DG framework which can be used in other settings as well. Our method allows for unstructured shape-regular meshes. We introduce a symmetric stabilization of the DG formulation as suggested in [25] for a linear hyperbolic problem.

The plan for this paper is as follows. In Section 2, we make precise some model assumptions. In some instances, (1) allows for exact solutions; we collect some of these. In Section 3 we introduce a DG method for this system, and analyze its stability properties. A key *a priori* estimate is presented in Section 4. Since the model is nonlinear, we provide details of the implementation in Section 5. Finally, in Section 6 we present some illustrative numerical examples supporting the results of Section 4.

2. Model problem

Let $\Omega = [0, \hat{A}] \times [0, T] \subset \mathbb{R}^2$ and set $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]^{\mathrm{T}}$ with $\alpha_1, \alpha_2 \in C^1(\overline{\Omega})$ being strictly positive. In addition, define

$$\Gamma^- = \{(a, t) \in \Gamma : \boldsymbol{\alpha}(a, t) \cdot \boldsymbol{n}_{\partial\Omega} < 0\} \tag{2}$$

and

$$\Gamma^+ = \{(a, t) \in \Gamma : \boldsymbol{\alpha}(a, t) \cdot \boldsymbol{n}_{\partial\Omega} > 0\} \tag{3}$$