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Analytic regularity and collocation approximation for elliptic PDEs with random domain deformations

Julio E. Castrillón-Candás[a,*,1], Fabio Nobile[b], Raúl F. Tempone[c,a]

[a] SRI Center for Uncertainty Quantification, 4700, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia

[b] CSQI-MATHICSE, Ecole Politechnique Federale Lausanne, Station 8, CH1015, Lausanne, Switzerland

[c] Applied Mathematics and Computational Science, 4700, King Abdullah University of Science and Technology, Thuwal, 23955-6900, Saudi Arabia

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ABSTRACT

In this work we consider the problem of approximating the statistics of a given Quantity of Interest (QoI) that depends on the solution of a linear elliptic PDE defined over a random domain parameterized by N random variables. The elliptic problem is remapped onto a corresponding PDE with a fixed deterministic domain. We show that the solution can be analytically extended to a well defined region in $\mathbb{C}^N$ with respect to the random variables. A sparse grid stochastic collocation method is then used to compute the mean and variance of the QoI. Finally, convergence rates for the mean and variance of the QoI are derived and compared to those obtained in numerical experiments.

1. Introduction

In many physical processes the practicing engineer or scientist encounters the problem of optimal design under uncertainty of the underlying domain. For example, in graphene sheet nano fabrication the exact geometries of the designed patterns (e.g. nano pores) are not easy to control due to uncertainties. If there is no quantitative understanding in the involved domain uncertainty such a design may be carried out by trial and error. However, in order to accelerate the design cycle, it is essential to quantify the influence of this uncertainty on *Quantities of Interest(QoI)*, for example, the sheet stress of the graphene sheet. Other examples include lithographic process introduced in semi-conductor design [1].

Collocation and perturbation approaches have been suggested in the past as an approach to quantify the statistics of the QoI with random domains [1–5]. The collocation approaches proposed in [2–4] work well for large amplitude domain perturbations although suffer from the curse of dimensionality. Moreover, these works lack error estimates of the QoI with respect to the number of sparse grid points. On the other hand, the perturbation approaches introduced in [5,1] are efficient for small variations of the domain.

In this paper we give a rigorous convergence analysis of the collocation approach based on isotropic Smolyak grids. This consists of an analysis of the regularity of the solution with respect to the parameters describing the domain perturbation.

* Corresponding author.
E-mail addresses: uvel@alum.mit.edu (J.E. Castrillón-Candás), fabio.nobile@epfl.ch (F. Nobile), raul.tempone@kaust.edu.sa (R.F. Tempone).

1 Also at 1758 Commonwealth Ave #10 Brighton, MA 02135. Tel.: +1 617 943 9981.

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In this respect we show that the solution can be analytically extended to a well defined region in $\mathbb{C}^N$ with respect to the random variables. Moreover, we derive error estimates both in the "energy norm" as well as on functionals of the solution (Quantity of Interest) for Clenshaw–Curtis abscissas that can be easily generalized to a larger class of sparse grids.

The outline of the paper is as follows: In Section 2 we set up the mathematical problem and reformulate the random domain elliptic PDE problem onto a deterministic domain with random matrix coefficients. We assume that the random boundary is parameterized by N random variables. In Section 3 we show that the solution can be analytically extended into a well defined region in $\mathbb{C}^N$. Theorem 7 is the main result of this paper. In Section 4 we set up the stochastic collocation problem and summarize several known sparse grid approaches that are used to approximate the mean and variance of the QoI. In Section 5 we assume that the random domain is truncated to $N_s \leqslant N$ random variables. We derive error estimates for the mean and variance of the QoI with respect to the finite element, sparse grid and truncation approximations. Finally, in Section 7 numerical examples are presented.

2. Setup and problem formulation

Let Ω be the set of outcomes from the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is a sigma algebra of events and $\mathbb{P}$ is a probability measure. Define $L^q_P(\Omega)$, $q \in [1, \infty]$, as the space of random variables such that

$$L^q_P(\Omega) := \left\{ v \mid \int_\Omega |v(\omega)|^q \, d\mathbb{P} < \infty \right\} \quad \text{and} \quad L^\infty_P(\Omega) := \{ v \mid \operatorname*{ess\,sup}_{\omega \in \Omega} |v(\omega)| < \infty \},$$

where $v : \Omega \to \mathbb{R}$ is a measurable random variable.

Suppose $D(\omega) \subset \mathbb{R}^d$ is an open bounded domain with Lipschitz boundary $\partial D(\omega)$ parameterized with respect to a stochastic parameter $\omega \in \Omega$. The strong form of the problem we consider in this work is: given sufficiently smooth regularity on $f(\cdot, \omega)$, $a(\cdot, \omega) : D(\omega) \to \mathbb{R}^d$, find $u(\cdot, \omega) : D(\omega) \to \mathbb{R}$ such that almost surely

$$\begin{aligned} -\nabla \cdot (a(x, \omega) \nabla u(x, \omega)) &= f(x, \omega), \qquad x \in D(\omega), \\ u &= 0 \qquad\qquad\;\; \text{on } \partial D(\omega). \end{aligned}$$

Now, assume the diffusion coefficient satisfies the following assumption.

Assumption 1. There exist constants a_{min} and a_{max} such that

$$0 < a_{min} \leqslant a(x, \omega) \leqslant a_{max} < \infty \quad \text{for a.e. } x \in D(\omega), \omega \in \Omega,$$

where

$$a_{min} := \operatorname*{ess\,inf}_{x \in D(\omega), \omega \in \Omega} a(x, \omega) \quad \text{and} \quad a_{max} := \operatorname*{ess\,sup}_{x \in D(\omega), \omega \in \Omega} a(x, \omega).$$

We now state the weak formulation as:

Problem 1. Find $u(\cdot, \omega) \in H^1_0(D(\omega))$ s.t.

$$\int_{D(\omega)} a(x, \omega) \nabla u(x, \omega) \cdot \nabla v(x) \, dx = \int_{D(\omega)} f(x, \omega) v(x) \, dx \quad \forall v \in H^1_0(D(\omega)) \text{ a.s. in } \Omega, \tag{1}$$

where $f(\cdot, \omega) \in L^2(D(\omega))$ for a.e. $\omega \in \Omega$.

Under Assumption 1 the weak formulation has a unique solution up to a zero-measure set in Ω.

2.1. Reformulation onto a fixed domain

Now, assume that given any $\omega \in \Omega$ the domain $D(\omega)$ can be mapped to an open and bounded reference domain $U \subset \mathbb{R}^d$ with Lipschitz boundary through a random map $F(\omega) : U \to D(\omega)$, where we assume that $F(\omega)$ is one-to-one and the determinant of the Jacobian $|\partial F(\cdot, \omega)| \in L^\infty(U)$ almost surely. Furthermore, we assume that $|\partial F|$ is uniformly greater than zero almost surely. We will, however, make the following equivalent assumption.

Assumption 2. Suppose that the map $F(\omega) : U \to D(\omega)$ is one-to-one a.s. and that there exist constants $\mathbb{F}_{min}$ and $\mathbb{F}_{max}$ such that

$$0 < \mathbb{F}_{min} \leqslant \sigma_{min}(\partial F(\omega)) \quad \text{and} \quad \sigma_{max}(\partial F(\omega)) \leqslant \mathbb{F}_{max} < \infty$$

almost everywhere in U and almost surely in Ω. We have denoted by $\sigma_{min}(\partial F(\omega))$ (and $\sigma_{max}(\partial F(\omega))$) the minimum (respectively maximum) singular value of the Jacobian $\partial F(\omega)$.