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Linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations

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ABSTRACT

Linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations are analyzed. Necessary and sufficient conditions are presented to apply the linear superposition principles. Applications of an algorithm using weights are made together with a few concrete examples.

1. Introduction

There has been a great interest in finding exact solutions to nonlinear differential equations in mathematical physics. Exact solutions help us understand the mechanism that governs physical phenomena in plasma physics, optical fibers, biology, solid state physics, chemical physics, and others [1–7]. Integrability theory of nonlinear partial differential equations tells how and when exact solutions can be obtained [8–10]. Due to the nonlinearity of differential equations, it is often difficult to present exact solutions to nonlinear PDEs.

The Hirota bilinear technique is a powerful tool to investigate integrability of differential equations and it is applied to many integrable equations including integrable couplings by perturbation [11], for which N-soliton solutions are obtained [12–14]. The existence of N-soliton solutions often implies the integrability of differential equations by quadratures. Wronskian and Casoratian solutions [15–19] and quasi-periodic solutions [20–22] can also be presented systematically based on Hirota bilinear forms. In [23], linear superposition principles of hyperbolic and trigonometric functions solutions to Hirota bilinear equations were analyzed and specific classes of N-soliton solutions were constructed, following studies on linear superposition principles [24,25].

In this paper, we would like to find necessary and sufficient conditions to guarantee existence of linear subspaces of hyperbolic and trigonometric function solutions to generalized bilinear equations. Generalized bilinear equations were introduced by adopting a new way of assigning symbols for derivatives [26,27]. Based on an equality established in [27], we will present a condition, being sufficient and necessary, for the linear superposition principle of hyperbolic and trigonometric function solutions.

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The rest of this paper is arranged as follows. In Section 2, we study the linear superposition principle of hyperbolic and trigonometric function solutions, aiming to construct a specific class of N-wave solutions. In Section 3, a few illustrative examples will be computed, as applications of the linear superposition principle established in the previous section. Finally, some conclusions will be provided.

2. Linear superposition principle

Let us begin with a bilinear equation with generalized bilinear derivatives:

$$P\left(D_{p,x_1}, D_{p,x_2}, \ldots, D_{p,x_M}\right) f \cdot f = 0, \tag{2.1}$$

where P is a polynomial in the indicated variables and $D_{p,x_i},\ 1 \leq i \leq M$, are generalized differential operators [26,27] defined by

$$\left(D_{p,x}^n f \cdot g\right)(x) = \left(\partial_x + \alpha \partial_{x'}\right)^n f(x) g(x')\big|_{x'=x} = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \left(\partial_x^{n-i} f\right)(x) \left(\partial_x^i g\right)(x), \quad n \geq 1, \tag{2.2}$$

in which the powers of α are determined by

$$\alpha^i = (-1)^{r(i)}, \quad \text{where } i = r(i) \bmod p \text{ with } 0 \leq r(i) < p,\ i \geq 0. \tag{2.3}$$

Now introduce N wave variables:

$$\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M, \quad 1 \leq i \leq N, \tag{2.4}$$

and exponential wave functions

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M}, \quad 1 \leq i \leq N, \tag{2.5}$$

where the $k_{j,i}$'s are real constants, and a wave related vector $\mathbf{k}_i$ and the dependent variable vector $\mathbf{x}$ are

$$\mathbf{k}_i = \left(k_{1,i}, k_{2,i}, \ldots, k_{M,i}\right), \quad 1 \leq i \leq N, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_M). \tag{2.6}$$

Take a linear combination

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{\eta_i}, \tag{2.7}$$

where $\varepsilon_i,\ 1 \leq i \leq N$, are arbitrary constants. It is known [27] that a linear combination f of N exponential wave solves a generalized bilinear equation (2.1) if and only if the following condition

$$P\left(k_{1,i} + \alpha k_{1,j}, \ldots, k_{M,i} + \alpha k_{M,j}\right) + P\left(k_{1,j} + \alpha k_{1,i}, \ldots, k_{M,j} + \alpha k_{M,i}\right) = 0, \quad 1 \leq i \leq j \leq N, \tag{2.8}$$

is satisfied.

2.1. Linear superposition principle of hyperbolic function solutions

We take $f_i = ch\eta_i = \frac{1}{2}\left(e^{\eta_i} + e^{-\eta_i}\right),\ 1 \leq i \leq N$, be hyperbolic function solutions to (2.1). Set

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i ch\eta_i = \sum_{i=1}^{N} \varepsilon_i \frac{1}{2}\left(e^{\eta_i} + e^{-\eta_i}\right), \tag{2.9}$$

which is a general linear combination of hyperbolic function solutions. The following identity holds for exponential functions under generalized bilinear derivatives [27]:

$$P\left(D_{p,x_1}, \ldots, D_{p,x_l}\right) e^{\eta_i} \cdot e^{\eta_j} = P\left(k_{1,i} + \alpha k_{1,j}, \ldots, k_{M,i} + \alpha k_{M,j}\right) e^{\eta_i + \eta_j}, \quad 1 \leq i,\ j \leq N. \tag{2.10}$$

Based on (2.10), we can compute that

$$\begin{aligned}
&P\left(D_{p,x_1}, \ldots, D_{p,x_M}\right) f \cdot f \\
&= P\left(D_{p,x_1}, \ldots, D_{p,x_M}\right) \sum_{i=1}^{N} \varepsilon_i ch\eta_i \cdot \sum_{j=1}^{N} \varepsilon_j ch\eta_j \\
&= \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P\left(D_{p,x_1}, \ldots, D_{p,x_M}\right) \frac{1}{2}\left(e^{\eta_i} + e^{-\eta_i}\right) \cdot \frac{1}{2}\left(e^{\eta_j} + e^{-\eta_j}\right)
\end{aligned}$$