Computers and Mathematics with Applications 71 (2016) 1287–1302

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Fourth order finite difference schemes for time–space fractional sub-diffusion equations

Hong-Kui Pang [a], Hai-Wei Sun [b,*]

[a] *School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, China*

[b] *Department of Mathematics, University of Macau, Macao, China*

ARTICLE INFO

Article history:
Received 14 June 2015
Received in revised form 4 February 2016
Accepted 10 February 2016
Available online 28 February 2016

Keywords:
Time–space fractional sub-diffusion equations
Fractional derivative
Fourth order finite-difference approximation
$L1$ approximation
Stability
Convergence

ABSTRACT

In this paper, we devote to the study of high order finite difference schemes for one- and two-dimensional time–space fractional sub-diffusion equations. A fourth order finite difference scheme is invoked for the spatial fractional derivatives, and the $L1$ approximation is applied to the temporal fractional parts. For the two-dimensional case, an alternating direction implicit scheme based on $L1$ approximation is proposed. The stability and convergence of the proposed methods are studied. Numerical experiments are performed to verify the effectiveness and accuracy of the proposed difference schemes.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the past decades, fractional calculus has gained great popularity due to its widespread applications in fields of science and engineering [1–5]. One of its most important applications is to describe the sub-diffusion and super-diffusion process [6–9]. The suitable mathematical models are the time and/or space fractional diffusion equations, where the classical first order derivative in time is replaced by the Caputo fractional derivative [10] of order $\gamma \in (0, 1)$, and the second order derivative in space is essentially replaced by the Riemann–Liouville fractional derivative [10] of order $\alpha \in (1, 2)$. It is well known that the analytical solutions to the fractional differential equations are usually difficult to derive and always contain some infinite series even if it is luckily obtained, which make evaluation very expensive. Therefore, the development of numerical methods for these problems has received enormous attention and undergone a fast evolution in recent years [11,6,12–16,9,17–24].

Among a variety of techniques developed for fractional differential equations, the finite difference method should be the most popular one because it is direct and convenient to use. Meerschaert and Tadjeran [18] initially proposed a shifted Grünwald–Letnikov discretization to approximate the space fractional differential equations with a left sided Riemann–Liouville fractional derivative, which they showed to be stable and first-order accuracy in space. Extensions of this scheme to address various space fractional differential equations [25,19,23,26] followed soon after. Recently, Deng and his co-workers [27–29] exploited the weighted and shifted skill to construct a series of high-order finite difference schemes,

* Corresponding author.
E-mail addresses: panghongkui@163.com (H.-K. Pang), HSun@umac.mo (H.-W. Sun).

http://dx.doi.org/10.1016/j.camwa.2016.02.011

named weighted and shifted Grünwald difference (WSGD) approximations, to the Riemann–Liouville space fractional derivatives. Motivated by this idea, Hao, Sun, and Cao [30] proposed a fourth-order quasi-compact difference scheme by carefully weighting the Grünwald approximation formula with different shifts and combining the compact technique for solving space fractional differential equations. For time fractional differential equations, the $L1$ formula [31] should be the major numerical differentiation formula to directly discretize the temporal fractional derivatives. Based on the $L1$ approximation, many stable numerical schemes have been established and analyzed [32–34,13,35,36] in the past decade.

Besides time or space fractional differential equations, the fractional differential equations with both temporal and spatial fractional derivatives have also received an increasing attention in recent years and have been used to model a wide range of phenomenons [37–42,21,22,43]. Therefore, the design of efficient and stable numerical schemes for time–space fractional differential equations is also an important activity. Liu et al. [44] proposed an implicit finite difference approximation to the time–space fractional diffusion equation, where the unconditional stability and first-order accuracy in both time and space were proved. Yang et al. [45] derived a novel numerical method based on the matrix transfer technique in space and finite difference scheme (or Laplace transform) in time to deal with the time–space fractional diffusion equations in two dimensions. Chen, Deng, and Wu [6] applied the $L1$ approximation to the time fractional derivative and second-order finite difference discretizations to the space fractional derivative for solving the two-dimensional time–space Caputo–Riesz fractional diffusion equation with variable coefficients in a finite domain. Most recently, an alternating direction implicit (ADI) scheme with second-order accuracy in both time and space is constructed to the time–space Caputo–Riesz fractional diffusion-wave equation by Wang, Vong, and Lei [46]. In the paper they also considered the time–space fractional sub-diffusion equation and constructed a full discretization difference scheme without dimensional (directional) splitting by the $L1$ formulae in time and second-order approximation in space.

In this paper, we focus on the high-order finite difference schemes for time–space fractional sub-diffusion equations. The proposed schemes are based on using the fourth-order quasi-compact difference scheme proposed by Hao, Sun, and Cao [30] for spatial approximation, which needs fewer grid points to produce a high accuracy solution. For the temporal discretization, we adopt the $L1$ approximation. Both one- and two-dimensional time–space fractional diffusion equations are considered. For the two-dimensional case, we also construct an ADI scheme based on the $L1$ approximation to reduce the storage requirement and the computational burden. Theoretical analyses show that the proposed schemes for both one- and two-dimensional cases are unconditionally stable and convergent.

The paper is organized as follows. In Section 2, we introduce the approximation of fourth-order quasi-compact finite difference scheme for Riemann–Liouville fractional derivatives and the $L1$ approximation to Caputo fractional derivatives. In Section 3, we apply these approximations to construct a full discretization scheme for the one-dimensional time–space fractional diffusion equation. The stability and convergence of the proposed scheme are discussed. In Section 4, we extend the discretization scheme to two-dimensional case. An ADI scheme based on $L1$ approximation is derived and the stability and convergence of the scheme are rigorously proved. Numerical examples are presented in Section 5 to support our theoretical analysis. Finally, concluding remarks are offered in Section 6.

2. Finite difference approximations of spatial and temporal fractional derivatives

We first introduce some definitions of fractional derivatives and then present their finite difference approximations.

Definition 2.1 (*[10]*)**.** For $\alpha \in (n-1, n)$ $(n \in \mathbb{N}^+)$, let $u(x)$ be $(n-1)$-times continuously differentiable on (a, ∞) (or $(-\infty, b)$ corresponding to the right derivative) and its n-times derivative be integrable on any subinterval of $[a, \infty)$ (or $(-\infty, b]$ corresponding to the right derivative). Then the left and right Riemann–Liouville fractional derivatives of the function $u(x)$ are defined as

$$ {}_a\mathcal{D}_x^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-n+1}} \mathrm{d}\xi $$

and

$$ {}_x\mathcal{D}_b^{\alpha} u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-n+1}} \mathrm{d}\xi, $$

respectively.

We remark that the 'a' in the definition can be extended to be '$-\infty$' and 'b' to be '$+\infty$'. In the following discussion, we assume that $u(x)$ is defined on $[a, b]$ and whenever necessary $u(x)$ can be smoothly zero extended to $(-\infty, b)$ or $(a, +\infty)$ or even $(-\infty, +\infty)$.

Definition 2.2 (*[10]*)**.** For $\gamma \in (n-1, n)$ $(n \in \mathbb{N}^+)$, let $u(t)$ be $(n-1)$-times continuously differentiable on $(0, \infty)$ and its n-times derivative be integrable on any subinterval of $[0, \infty)$. Then the Caputo fractional derivative of the function $u(t)$ is defined as

$$ {}_0^C\mathcal{D}_t^{\gamma} u(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{u^{(n)}(\zeta)}{(t-\zeta)^{\gamma-n+1}} \mathrm{d}\zeta, \quad t \in (0, \infty). $$