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Compact difference scheme for a class of fractional-in-space nonlinear damped wave equations in two space dimensions☆

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ABSTRACT

In this paper, we focus our attention on the numerical solution of a class of fractional-in-space nonlinear damped wave equations in two space dimensions. The fractional-in-space telegraph equation, sine–Gordon equation and Klein–Gordon equation can be regarded as particular cases of such equations. A compact difference scheme with accuracy of fourth-order in space and second-order in time is proposed. The solvability, stability and convergence of the scheme are shown under a common assumption. In order to reduce the computational burden, a compact alternating direction implicit (ADI) difference scheme is established. An ADI scheme with accuracy of second-order in both time and space is also derived. Both ADI schemes are applied to solve the aforesaid three kinds of equations. Numerical results are provided to verify the theoretical analysis.

1. Introduction

In recent decades, physicists, mechanics and engineers have realized that more and more phenomena in natural systems cannot be described well via conventional differential equations, such as turbulence [1], hydrology [2], anomalous diffusion [3] and stock movement [4]. In view of the fact that the fractional calculus possesses the ability to depict the inheritance and memory features, it has been tried to simulate the phenomena above. A mass of experimental results indicate that the attempt is successful [5,6]. As a result, a variety of fractional differential models have been proposed by replacing the integral-order time or space differential operators in traditional differential models with the fractional ones.

In this paper, we consider the difference solution of high accuracy for solving fractional-in-space nonlinear damped wave equations in two space dimensions, which has the general form

$$\frac{\partial^2 u}{\partial t^2} + \rho \frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^\alpha u}{\partial |x|^\alpha} + \frac{\partial^\alpha u}{\partial |y|^\alpha} \right) + f(u, x, y, t), \quad (x, y) \in \Omega,\ 0 < t \le T, \tag{1.1}$$

with the initial conditions

$$u(x, y, 0) = \phi(x, y), \qquad \frac{\partial u}{\partial t}(x, y, 0) = \psi(x, y), \quad (x, y) \in \overline{\Omega} = \Omega \cup \partial\Omega, \tag{1.2}$$

and the boundary condition

$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega,\ 0 < t \le T, \tag{1.3}$$

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where $1 < \alpha \le 2, \rho \ge 0, \kappa > 0, \Omega = (a, b) \times (c, d)$ and $\partial\Omega$ is the boundary of Ω. The functions $\phi(x, y)$, $\psi(x, y)$ and $f(u, x, y, t)$ are supposed to be sufficiently smooth to ensure consistency and convergence rate of the difference scheme under consideration. The Riesz fractional derivative with order α is defined by

$$\frac{\partial^\alpha u}{\partial |x|^\alpha}(x, y, t) = -\frac{1}{2\cos(\pi\alpha/2)}({}_aD_x^\alpha + {}_xD_b^\alpha)u(x, y, t), \tag{1.4}$$

where

$${}_aD_x^\alpha u(x, y, t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x \frac{u(\eta, y, t)}{(x-\eta)^{\alpha-1}}, \qquad {}_xD_b^\alpha u(x, y, t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\eta, y, t)}{(\eta-x)^{\alpha-1}} \tag{1.5}$$

are the left and right Riemann–Liouville fractional derivatives in x-direction [7]. $\partial^\alpha/\partial|y|^\alpha$ is defined similarly. When $\alpha = 2$, $\partial^\alpha/\partial|x|^\alpha + \partial^\alpha/\partial|y|^\alpha$ reduces to the classical two-dimensional Laplacian operator, the corresponding problems have been widely studied, see e.g., [8,9].

The model (1.1) occurs in various physical situations. It covers a series of important fractional partial differential equations, such as the fractional-in-space telegraph equation [10,11], Cattaneo equation [12], Klein–Gordon equation [13] and sine–Gordon equation [14,15]. The numerical methods for solving the equations with time-fractional derivative can be found in [16–19]. The above-mentioned equations have great significance in signal analysis, thermodynamics, quantum mechanics, etc. The integer-order telegraph equation, Cattaneo equation, Klein–Gordon equation and sine–Gordon equation have been extensively studied from the theoretical and numerical aspects, see e.g., [20–25]. Because of the non-local property of fractional derivative, it is more difficult to solve numerically the corresponding fractional differential equations. Therefore, the development of excellent numerical methods for the problem (1.1)–(1.3) is of great importance, especially for high-dimensional situations.

The key to develop numerical algorithms with high accuracy for fractional differential equations is the high-order approximation to fractional derivative. In [26], Meerschaert and Tadjeran proposed the shifted Grünwald–Letnikov discretization for the left and right Riemann–Liouville fractional derivatives. This approach is subsequently applied to the two-sided fractional-in-space diffusion equation [27] and the two-dimensional nonlinear fractional-in-space diffusion equation [28]. Nevertheless, the shifted Grünwald–Letnikov formula only has first-order accuracy. In order to obtain the approximation with higher order accuracy, Ortigueira [29] derived the fractional centered difference formula of second-order accuracy for the Riesz fractional derivative. Çelik and Duman [30] analyzed in detail the approximation error of the second-order formula, and then applied it to the fractional-in-space diffusion equation. Later more research works are published by utilizing this discretization. For instance, Wang et al. [31] and Ran et al. [32] applied it to fractional-in-space Schrödinger equations with different forms, in which some conservative difference schemes are proposed. Hou et al. [33] derived two difference schemes for solving the fractional-in-space Allen–Cahn equation with small perturbation parameter and strong nonlinearity. Each scheme is shown to preserve the discrete maximum principle and the energy decaying properties under certain restrictions on the time increment. In [34], Tian et al. introduced the weighted and shifted Grünwald difference operator with accuracy of second-order, and then established a second-order difference scheme for the fractional-in-space diffusion equations. Based on this idea of the paper [35], Chen and Deng [36] presented the weighted and shifted Lubich difference operator with fourth-order accuracy, and applied it to solve the fractional-in-space diffusion equations with variable coefficients in one and two dimensions. Very recently, Ding et al. [37], Zhao et al. [38] and Hao et al. [39] developed the fourth-order approximations for the Riesz fractional derivative and the left and right Riemann–Liouville fractional derivatives on the basis of the fractional centered difference formulae and the shifted Grünwald formulae.

The main novelty of this paper is that we construct a fourth-order compact difference scheme for the two-dimensional fractional-in-space nonlinear damped wave equations (1.1)–(1.3). Using energy method, we demonstrate the solvability, convergence and stability of the scheme under a common assumption. A compact ADI difference scheme is constructed to reduce the computational expense. An ADI scheme with accuracy of second-order in both time and space is also given.

The rest of the paper is organized as follows. In Section 2, we introduce some necessary notations and a lemma, and then derive the compact difference scheme. In Section 3, using energy method, the solvability and convergence of the compact scheme are proved. In Section 4, a compact ADI scheme is constructed by adding a small term, and an ADI scheme with accuracy of second-order in both time and spaces is mentioned. In the last section, both of the ADI schemes are applied to solve the two-dimensional fractional-in-space telegraph equation, sine–Gordon equation and Klein–Gordon equation, and some relevant numerical results are reported to support the theoretical analysis. The paper ends with a brief summary of conclusions.

2. Compact difference scheme

For a positive integer N, let $t_n = n\tau (0 \le n \le N)$ and $\Omega_\tau = \{t_n | 0 \le n \le N\}$, where $\tau = T/N$ is the uniform temporal step size. For any grid function $v = \{v^n | 0 \le n \le N\}$, denote

$$\delta_t v^{n+\frac{1}{2}} = \frac{1}{\tau}(v^{n+1} - v^n), \qquad \delta_t^2 v^n = \frac{1}{\tau}\left(\delta_t v^{n+\frac{1}{2}} - \delta_t v^{n-\frac{1}{2}}\right),$$