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Global existence and blow-up of solutions for a Non-Newton polytropic filtration system with special volumetric moisture content ☆

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ABSTRACT

This paper deals with a doubly degenerate parabolic system with special volumetric moisture content, which is called a Non-Newton polytropic filtration system. Under appropriate hypotheses, we prove that the solution either exists globally or blows up in finite time.

1. Introduction

Suppose a compressible fluid flows in a homogeneous isotropic rigid porous medium. Then the volumetric moisture content $\theta(x)$, the macroscopic velocity $\vec{V}$ and the density of the fluid ρ are governed by the following equation [1]:

$$\theta(x)\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho\vec{V}\right) - f(\rho) = 0, \tag{1.1}$$

where $f(u)$ is the source. For the non-Newtonian fluid, the linear Darcy's law is no longer valid, because the influence of many factors such as the molecular and ion effects need to be concerned. Instead, one has the following nonlinear relation

$$\rho\vec{V} = -\lambda\,|\nabla P|^{\alpha-2}\,\nabla P, \tag{1.2}$$

where $\rho\vec{V}$ and P denote the momentum velocity and pressure respectively, $\lambda > 0$ and $\alpha \geq 2$ are some physical constants.

If the fluid considered is the polytropic gas, then the pressure and density satisfy the following equation of the state

$$P = c\rho^{\gamma}, \tag{1.3}$$

where $c > 0$, $\gamma > 0$ are some constants. Thus, from (1.1)–(1.3), we get

$$\theta(x)\frac{\partial \rho}{\partial t} = c^{\alpha}\lambda \operatorname{div}\left(|\nabla\rho^{\gamma}|^{\alpha-2}\,\nabla\rho^{\gamma}\right) + f(\rho). \tag{1.4}$$

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During the last decade the above problem has enjoyed a growing attention. Much work has been done for $\theta(x) = 1$. Sattinger [2] constructed the so called stable set, which was used in order to construct global solutions (see [3–5]). Furthermore, with respect to the singularity properties, such as blowup, extinction and quenching, there are a lot of works in recent years (see [6–11,1,12–18] and references therein), and see also [19–24] for blow-up of solutions to parabolic equations with positive initial energy. However, much less effort has devoted to the case that $\theta(x)$ is not a constant. Tan [25] considers problem (1.4) with $\theta(x) = |x|^{-2}$ and $\gamma = 1$. By using the Hardy inequality and potential well (see [2]), he studied the global existence and blow up of the solutions. Wang [26] generated his results to the case $\theta(x) = |x|^{-\delta}$ with $0 \le \delta \le 2$ and $\gamma = 1$, and the case $\alpha = 2$ and $\gamma > 1$ was considered in [27]. However, to my knowledge, there is no paper to study the case $\alpha \ge 2$, $\gamma \ge 1$ and $\theta(x)$ is not a constant, which is the main task of this paper.

In this paper we consider (1.4) with $\theta(x) = |x|^{-\delta}$ and $f(\rho) = \rho^{\sigma}$. Furthermore, we incorporate zero boundary condition to this problem. Then we get the following initial–boundary problem after changing variables and notations:

$$\begin{cases} |x|^{-s}\dfrac{\partial u}{\partial t} - \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) = u^q, & (x,t) \in \Omega \times (0,T),\\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T),\\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N (N > p)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < T \le \infty$, $p \ge 2$, $m \ge 1$, $0 \le s \le 1 + 1/m \le 2$, and $mp - m < q \le \frac{m(Np-N+p)}{N-p}$.

Definition 1.1. We say that a nonnegative function $u(x,t)$ is a solution of problem (1.5) on $Q_T := \Omega \times (0,T)$ if

$$u^m \in L^\infty\left(0,T; W_0^{1,p}(\Omega)\right),\ u \in L^{m+q}(Q_T),\quad \iint_{Q_T} |x|^{-s}\left(\left(u^{(1+m)/2}\right)_t\right)^2 dxdt < \infty,$$

and $u(x,t)$ satisfies problem (1.5) in the distribution sense. Furthermore, we use $u(x,t;u_0)$ as the solution of (1.5) with an initial value $u_0(x)$.

For

$$u \in Q := \left\{u : u^m \in W_0^{1,p}(\Omega),\ u \in L^{m+q}(\Omega)\right\} \setminus \{0\}, \tag{1.6}$$

we define the energy functional $E(u)$ as follows:

$$E(u) = \frac{1}{mp}\int_\Omega \left|\nabla u^m\right|^p dx - \frac{1}{m+q}\int_\Omega |u|^{m+q} dx. \tag{1.7}$$

Let

$$F_u(t) = E(tu) = \frac{t^{mp}}{mp}\int_\Omega \left|\nabla u^m\right|^p dx - \frac{t^{m+q}}{m+q}\int_\Omega |u|^{m+q} dx, \quad t \ge 0. \tag{1.8}$$

We define the *Nehari manifold* by

$$K := \left\{u \in Q : F_u'(1) = 0\right\}. \tag{1.9}$$

It is obvious that $u \in K$ if and only if $u \in Q$ and $H(u) = 0$, where

$$H(u) := \int_\Omega \left|\nabla u^m\right|^p dx - \int_\Omega |u|^{m+q} dx. \tag{1.10}$$

Further, we define the *Potential depth* by

$$d = \inf_{u\in Q}\left\{\sup_{t\ge 0} F_u(t)\right\}. \tag{1.11}$$

Next we calculate the value of d. To this end, we recall the Sobolev–Poincaré's inequality:

Lemma 1.2. *Let $mp - m < q \le \frac{m(Np-N+p)}{N-p}$. Then there exists a constant M depending only on Ω, m, N, p and q such that for all $u \in Q$ it holds*

$$\|u^m\|_{L^{\frac{m+q}{m}}(\Omega)} \le M\|\nabla u^m\|_{L^p(\Omega)}. \tag{1.12}$$

Furthermore, M is optimal in the sense of (1.12), i.e.,

$$\frac{1}{M} = \inf_{u\in Q}\left(\frac{\|\nabla u^m\|_{L^p(\Omega)}}{\|u^m\|_{L^{\frac{m+q}{m}}(\Omega)}}\right). \tag{1.13}$$