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Upper bound estimate for the blow-up time of an evolution m-Laplace equation involving variable source and positive initial energy[☆]

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ABSTRACT

In this paper, we study an evolution m-Laplace equation involving variable source and positive initial energy. We establish a blow-up result for a certain solution with positive initial energy and estimate the upper bound of the blow-up time for the blowing up solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following evolution m-Laplace equation involving variable source

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = |u|^{p(x)-1}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \not\equiv 0, & x \in \Omega \end{cases} \tag{1.1}$$

where $m > 1$ is a constant, the function $p(x) : \Omega \mapsto (1, +\infty)$ satisfies

(P_1) $1 < p^- := \min_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty$;
(P_2) $\forall \xi = (\xi_1, \ldots, \xi_N)$ and $\eta = (\eta_1, \ldots, \eta_N) \in \Omega$,

$$|\xi - \eta| := \sqrt{\sum_{i=1}^{n} |\xi_i - \eta_i|^2} < 1, \qquad |p(\xi) - p(\eta)| < \omega(|\xi - \eta|),$$

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where $\omega(\cdot) : [0, +\infty) \mapsto \mathbb{R}$ is a continuous function such that

$$\limsup_{\tau \to 0} \left| \omega(\tau) \ln \frac{1}{\tau} \right| = C < +\infty$$

and C is a constant;

(P_3) $1 < p^+ < +\infty$ with $N \le m$ or $1 < p^+ < \frac{Nm+m-N}{N-m}$ if $N > m$.

The equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$$

is called the non-Newtonian filtration equation, which is also called evolution m-Laplace equation (see [1,2]). Problems related to (1.1) arise in many mathematical models of applied science, such as nuclear science, chemical reactions, heat transfer, population dynamics, biological science, and have attracted a great deal of attention in the literature, see [3–7] and the references therein. Our main aim in this paper is to study a phenomenon that is called blow-up in the literature, which means that there exists a solution $u(x, t)$ which becomes unbounded in finite time. In recent years, blow-up has deserved a great interest, see for instance the books [8,9], the survey papers [10–12] and the references therein.

The blow-up results of (1.1) with $m = 2$ and $p(x)$ is a constant were first studied in [13], and numerous generalizations and modifications can be found in [14–18] and references therein. Problem (1.1) with $m > 1$ and $p(x)$ is a constant was studied in [19–21] and references therein, where blow-up and global existence conditions of solutions, blow-up rates and blow-up sets were got. The results on upper or lower bounds of the blow-up time were considered in [22,23]. Problem (1.1) with $m = 2$ and $p(x)$ is not a constant was studied in [24,25], where the critical exponent was characterized in [24] and the blow-up conditions for the problem with positive initial energy were established in [25].

However, up to our knowledge, it seems there is no paper where the blow-up phenomenon is studied with $m > 1$ and a variable exponent as a reaction term. So, it is natural to consider problem (1.1).

Next, we introduce some preliminaries and notations, which will be used throughout this paper. Set

$$L^{p(\cdot)} := \left\{ f \,\middle|\, f \text{ is a measurable function } \int_\Omega |f(x)|^{p(x)} dx < +\infty \right\}.$$

By [5, Theorem 3.2.7], the space $L^{p(\cdot)}$ equipped with the norm

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}$$

is a Banach space. It follows directly from definition that

$$\min\left\{ \|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+} \right\} \le \int_\Omega |f(x)|^{p(x)} dx \le \max\left\{ \|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+} \right\}. \tag{1.2}$$

By [5, Corollary 3.3.4], $L^{p^+ +1}(\Omega) \hookrightarrow L^{p(\cdot)+1}(\Omega)$ continuously. So it follows from (P_3) that the Sobolev space $W_0^{1,m}(\Omega) \hookrightarrow L^{p(\cdot)+1}(\Omega)$ continuously. Let B be the optimal constant of the embedding, then

$$\|u\|_{p(\cdot)+1} \le B \|\nabla u\|_m. \tag{1.3}$$

Let B_1 be a constant satisfying

$$B_1 \ge B \quad \text{and} \quad B_1 > 1. \tag{1.4}$$

$$\alpha_1 := B_1^{-\frac{p^- +1}{p^- +1-m}}, \tag{1.5}$$

$$E_1 := \frac{p^- + 1 - m}{m(p^- + 1)} B_1^{-\frac{m(p^- +1)}{p^- +1-m}} = \frac{p^- + 1 - m}{m(p^- + 1)} \alpha_1^m. \tag{1.6}$$

For $u(x, t)$ be the solution of problem (1.1), we define the energy functional as follows:

$$E(t) := \frac{1}{m} \int_\Omega |\nabla u(x, t)|^m dx - \int_\Omega \frac{1}{p(x) + 1} |u|^{p(x)+1} dx, \tag{1.7}$$

and denote $E(0)$ by

$$E(0) := \frac{1}{m} \int_\Omega |\nabla u_0(x)|^m dx - \int_\Omega \frac{1}{p(x) + 1} |u_0|^{p(x)+1} dx. \tag{1.8}$$

Then the main result of this paper reads as follows.