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Nonlocal problem for a general second-order elliptic operator

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ABSTRACT

The paper deals with a nonlinear second-order elliptic equation with Dirichlet boundary conditions. The nonlocal term involved in the strong problem essentially increases its complexity and the necessary total computational work. The existence and uniqueness of the weak solution is established. The nonlinear weak formulation is reduced to the minimization of a nonlinear functional. Finite element discretizations by Lagrangian finite elements are applied to obtain an approximate minimization problem. A two-point step size gradient method with an original steplength is used for finding approximate solutions of the problem under consideration. No line search is necessary in the new approach. The present method is computer implemented and tested on different triangulations. The test examples indicate that the method slightly depends on the initial guesses.

1. Introduction

The nonlocal nonlinear equations have been the object of a great interest in the recent decades [1–6] since they are widely applicable not only in engineering sciences but also in computational biology. Most of the authors consider quasilinear problems [7,8] of Kirchhoff type or semilinear problems [9,10]. Gudi [11] introduces a sparse Jacobian method for solving some nonlocal problems of Kirchhoff type. A more general reaction–diffusion equation is considered by Sanni [12]. The object of interest in the present paper is a nonlocal nonlinear reaction–diffusion problem. The investigation is carried out in the case of a general second-order elliptic operator. The existence and uniqueness of the weak solution is established. The major contribution of the present paper is the effective iterative method for solving the nonlocal nonlinear elliptic problem. The Barzilai–Borwein method is applied for solving a quartic minimization problem provided with an original steplength and no line search procedures. A Q-linear convergence is proved for the approximate solutions. The method with the new steplength is computer implemented. The numerical examples demonstrate that the initial guesses have no effect either on the convergence of the method or on the rate of convergence although, if the initial guesses are far away from the true solution, more iterations are necessary to get an acceptable approximation for the exact solution.

The rest of the paper is organized as follows. The problem of interest is defined in Section 2. The weak formulation is obtained in Section 3. In the same section the existence and uniqueness of the weak solution is proved. The weak formulation is reduced to a minimization problem for a nonlinear functional. Finite element discretization is introduced in Section 4. An iterative scheme for solving the discrete problem is compiled in Section 5. The Q-linear convergence is proved for the approximate solutions in the same section. Section 6 contains some numerical results supporting the considered theory. Concluding remarks are included in Section 7.

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2. Problem definition

Rewrite some basic definitions and denotations. Let Ω be an open smooth subset of $\mathbf{R}^2$. As usual, we denote the real Sobolev space $W_p^k(\Omega)$ for nonnegative integers k and $p = 2$ by $H^k(\Omega)$. The space $H^k(\Omega)$ is provided with the norm $\|\cdot\|_{k,\Omega}$ and the seminorm $|\cdot|_{k,\Omega}$. Denote the norm in $C^k(\overline{\Omega} \times \mathbf{R})$, $k \in \mathbf{N}$ by $|||\cdot|||_{k,\Omega}$. Introduce the norm

$$\|D^k F(x)\| = \sup_{\substack{\|\xi_i\|\le 1\\ 1\le i\le k}} \|D^k F(x)(\xi_1, \xi_2, \ldots, \xi_k)\|$$

for the k-th Fréchet derivative $D^kF(x)$. The L^2-scalar product is denoted by

$$(u, v) = \int_\Omega uv dx.$$

Define the space

$$\mathbf{V} = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}$$

and the second-order elliptic linear operator

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \text{a.u}, \qquad \text{dom}L = C_0^2\left(\overline{\Omega}\right),$$

where $a_{ij}(x)$ and $a(x)$ belong to $C^1(\overline{\Omega})$, $a_{ij} = a_{ji},\ i, j = 1, 2$ and $a(x) \ge a_0 > 0,\ \forall x \in \Omega$. Assume that L is uniformly elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$\alpha \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \quad \forall \xi, x \in \mathbf{R}^2. \tag{1}$$

Suppose that:

$$g \in C^1(\overline{\Omega} \times \mathbf{R}), \quad \frac{\partial g}{\partial u}(x, u) \ge 0,\ \forall x \in \overline{\Omega},\ g(x, 0) = 0, \tag{2}$$

$$f \in L^2(\Omega) \quad \text{with } \|f\|_{0,\Omega} \ne 0. \tag{3}$$

Consider the following nonlocal nonlinear elliptic problem

$$\mathcal{S}: \begin{cases} \text{Find } u \in C_0^2\left(\overline{\Omega}\right) \text{ satisfying:}\\ (Lu, u)\, Lu + g(x, u) = f(x) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

3. Weak formulation

The main goal in this section is to prove that the problem $(\mathcal{S})$ has a unique weak solution. For this purpose an associated minimization problem is considered. By applying Green's theorem to $(\mathcal{S})$, we obtain the weak formulation

$$\mathcal{W}: \begin{cases} \text{Find } u \in \mathbf{V} \text{ such that}\\ a(u, u)a(u, v) + (g(x, u), v) = (f, v) \quad \text{in } \Omega, \end{cases}$$

where $a(u, v)$ is the bilinear form

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx + \int_\Omega a(x)uv\, dx$$

and $(\cdot, \cdot)$ is the L^2-scalar product. Since L is a linear continuous and uniformly $\mathbf{V}$-elliptic operator, there exist positive constants $\underline{\alpha}$ and $\overline{\alpha}$ such that

$$\underline{\alpha}\|u\|_{1,\Omega}^2 \le a(u, u), \qquad a(u, v) \le \overline{\alpha}\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \quad \forall u, v \in \mathbf{V}. \tag{4}$$

Define the objective functional

$$J(v) = \frac{a^2(v, v)}{4} + B(v) - F(v),$$