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Mixed formulation for interface problems with distributed Lagrange multiplier

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ABSTRACT

We study a mixed formulation for elliptic interface problems which has been recently introduced when dealing with a test problem arising from fluid–structure interaction applications. The formulation, which involves a Lagrange multiplier defined in the solid domain, can be approximated by standard finite elements on meshes which do not need to fit with the interface. In this paper we discuss a modification of the original formulation involving a different approach for the analysis and the numerical implementation of the Lagrange multiplier. New two-dimensional numerical results confirm the good performances of the proposed schemes.

1. Introduction

It is very frequent that real world applications can be mathematically described by equations which contain as basic ingredient interface or transmission problems. By interface problem we mean a partial differential equation posed in a domain subdivided into two or more subdomains by interfaces where the coefficients may jump: the coefficient discontinuities introduce transmission conditions among the different subdomains.

The applications we have in mind arise from the approximation of fluid–structure interactions through the finite element Immersed Boundary Method (IBM) [1–3]. The main motivation for the present research actually comes from recent developments in the modeling of the IBM. It is out of the aims of this paper to describe the links in more detail, for which we refer the interested reader to [4,5]. In particular, we observe that the terminology IBM has been used in several (sometimes very different) contexts. Here we refer to the framework introduced by Peskin [1].

It is well known that the approximation of interface problems generally requires a mesh which is compatible with the interface in order to achieve optimal convergence rates. In the case of moving boundaries, this requires to adapt the mesh at each time step and, in any case, limits the shape of the admissible interfaces. A quite large literature deals with this problem and several possible workarounds have been proposed. Some basic references can be found, e.g., in [6–16].

We remark that the main goal of this research is not simply to develop an additional, perhaps more sophisticated, method for the approximation of interface problems, but is to present a stepping stone on the way to more significant modifications of the finite element IBM. The results of this paper, in particular, will be useful for proving the stability properties of a distributed Lagrangian version of the IBM, see [17]. For this reason, it is out of the aims of this paper to compare the performance and the efficiency of our method with other existing schemes.

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In [5] we have introduced a new scheme for the approximation of interface problems which makes use of a Lagrange multiplier and has some similarities with the fictitious domain approach with distributed Lagrange multiplier, see, e.g., [18,19]. The aim of this paper is, on the one hand, to present and analyze a new variant of the method and, on the other hand, to extend the preliminary one-dimensional numerical results of [5] to a more interesting two-dimensional case. The modification of the method consists in a different treatment of a duality pairing which in the original approximation was interpreted numerically as a scalar product in L^2. Here, using the Riesz identification, we propose to replace the duality pairing with the scalar product in H^1. From our numerical experiments it turns out that the new scheme is more robust, in the sense that it provides a good approximation for quite general coefficient jumps. We consider the case when two materials are present; in this case we use two meshes: one for the whole domain and one for the region where only one material is present. We emphasize that the meshes are completely independent from each other and, in particular, do not need to match along the interface between the two subdomains.

The results of the paper are supported by a rigorous theoretical analysis, which gives sufficient conditions for the convergence of the scheme in terms of the problem coefficients and the ratio between the size of the two meshes.

The problem is presented in Section 2, together with the variational formulations corresponding to the use of the Lagrange multiplier. In the same section the analysis of the mixed formulations is performed. In Section 3 the finite element approximation of both formulations is carried on and the error estimates are proved. The numerical tests, as well as the related results, are reported in Section 4.

2. Interface problem and its mixed formulations

Let $\Omega \subseteq \mathbb{R}^d,\ d = 1, 2, 3$, be a bounded domain with Lipschitz continuous boundary. We assume that there exists a Lipschitz continuous interface $\Gamma \subset \Omega$ which splits Ω into two subdomains Ω_1 and Ω_2. Let $\beta_1 : \Omega_1 \to \mathbb{R}$ and $\beta_2 : \Omega_2 \to \mathbb{R}$ be two bounded continuous functions such that $0 < \underline{\beta} \le \beta_i,\ i = 1, 2$. Let us consider the following interface problem: given $f_1 : \Omega_1 \to \mathbb{R}$ and $f_2 : \Omega_2 \to \mathbb{R}$, find $u_1 : \Omega_1 \to \mathbb{R}$ and $u_2 : \Omega_2 \to \mathbb{R}$ such that

$$
\begin{aligned}
&-\mathrm{div}(\beta_1 \nabla u_1) = f_1 \quad \text{in } \Omega_1\\
&-\mathrm{div}(\beta_2 \nabla u_2) = f_2 \quad \text{in } \Omega_2\\
&u_1 = u_2 \quad \text{on } \Gamma\\
&\beta_1 \nabla u_1 \cdot n_1 + \beta_2 \nabla u_2 \cdot n_2 = 0 \quad \text{on } \Gamma\\
&u_1 = 0 \quad \text{on } \partial\Omega_1 \setminus \Gamma\\
&u_2 = 0 \quad \text{on } \partial\Omega_2 \setminus \Gamma,
\end{aligned}
\tag{1}
$$

where n_1 and n_2 are the unit vectors normal to Γ, pointing outwards with respect to Ω_1 and Ω_2, respectively.

Let $H^1(\Omega_i)$ be the Hilbert space of real function in $L^2(\Omega_i)$ with gradient in $L^2(\Omega_i)^d,\ i = 1, 2$. Then we shall consider the following spaces:

$$
\begin{aligned}
&H^1_D(\Omega_i) = \{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega_i \setminus \Gamma\} \quad i = 1, 2,\\
&\mathbf{W} = \{(v_1, v_2) \in H^1_D(\Omega_1) \times H^1_D(\Omega_2) : v_1|_\Gamma = v_2|_\Gamma\}.
\end{aligned}
\tag{2}
$$

The above problem can be written in variational form as follows: find $(u_1, u_2) \in \mathbf{W}$ such that

$$\int_{\Omega_1} \beta_1 \nabla u_1 \nabla v_1\, dx + \int_{\Omega_2} \beta_2 \nabla u_2 \nabla v_2\, dx = \int_{\Omega_1} f_1 v_1\, dx + \int_{\Omega_2} f_2 v_2\, dx \quad \forall (v_1, v_2) \in \mathbf{W}. \tag{3}$$

The finite element discretization of (3) requires the construction of meshes in Ω_1 and Ω_2 which fit with the interface Γ. Moreover, in order to enforce easily the continuity at the boundary of the two components one should choose matching grids so that the meshes share the nodes on Γ. If the interface problem above arises as a step in an advancing scheme in the resolution of fluid–structure interaction problems, the interface Γ might depend on time so that at each time step one has to change the mesh close to the interface. To avoid this inconvenience we have proposed in [5] the following *fictitious formulation with a distributed Lagrange multiplier*.

We set $\Omega = \Omega_1 \cup \Omega_2$. Let $v \in H^1_0(\Omega)$, then its restrictions $v_1 = v|_{\Omega_1}$ and $v_2 = v|_{\Omega_2}$ are such that the pair (v_1, v_2) belongs to $\mathbf{W}$. Let us consider extensions $\beta : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ of β_1 and f_1, such that $\beta|_{\Omega_1} = \beta_1$ and $f|_{\Omega_1} = f_1$ and β is continuous on Ω.

Let $u \in H^1_0(\Omega)$ be such that $u|_{\Omega_1} = u_1$ and $u|_{\Omega_2} = u_2$, then we can rewrite Eq. (3) as

$$\int_\Omega \beta \nabla u \nabla v\, dx + \int_{\Omega_2} (\beta_2 - \beta) \nabla u_2 \nabla v|_{\Omega_2}\, dx = \int_\Omega f v\, dx + \int_{\Omega_2} (f_2 - f) v|_{\Omega_2}\, dx \quad \forall v \in H^1_0(\Omega).$$

The above equation is equivalent to (3) if and only if $u|_{\Omega_2} = u_2$. Introducing a Lagrange multiplier associated to this constraint we arrive to the following problem. We use the following notation: $V = H^1_0(\Omega)$ and $V_2 = H^1(\Omega_2)$ and $\Lambda = [H^1(\Omega_2)]^*$, that is the dual space of $H^1(\Omega_2)$. These spaces are endowed with their natural norms.