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Finite element approximations of general fully nonlinear second order elliptic partial differential equations based on the vanishing moment method

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ABSTRACT

The vanishing moment method was introduced by the authors in Feng and Neilan (2009) as a reliable methodology for computing viscosity solutions of fully nonlinear second order partial differential equations (PDEs). It is based on the simple idea of approximating a "hard-to-handle" fully nonlinear second order PDE by a family (parameterized by a small parameter ε) of "easy-to-handle" quasilinear fourth order PDEs. The primary objective of this article is to present a comprehensive finite element analysis for the vanishing moment approximation of general fully nonlinear second order elliptic PDEs which fulfill some structure conditions. Abstract methodological and convergence analysis frameworks of conforming finite element methods are first developed for fully nonlinear second order PDEs in a general setting. The abstract framework is then applied to three prototypical nonlinear equations, namely, the Monge–Ampère equation, the equation of prescribed Gauss curvature, and the infinity-Laplacian equation. Numerical experiments are presented for each problem to validate the theoretical error estimate results and to gauge the efficiency of the proposed numerical methods and the vanishing moment methodology.

1. Introduction

This paper concerns finite element approximations of general fully nonlinear second order elliptic PDEs of the following form:

$$F(D^2u, \nabla u, u, x) = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.1b}$$

where D^2u and ∇u represent the Hessian and the gradient of u, respectively. $g : \partial\Omega \to \mathbb{R}$ and $F : \mathcal{S}^{d\times d} \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ are continuous functions, and F is assumed to be elliptic and nonlinear in at least one of its entries of D^2u. Further structure assumptions on F will be given in the subsequent section. Fully nonlinear PDEs (1.1) arise from many scientific and engineering fields including differential geometry, optimal control, mass transportation, geostrophic fluid, meteorology, and general relativity (cf. [1–6] and the references therein). Although numerical methods for special equations (e.g., the Monge–Ampère equation) of the form (1.1) have been recently developed in the literature (cf. [7–11] and the references therein), little work has been reported for general fully nonlinear second order PDEs in high dimensions.

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Because of the full nonlinearity in (1.1), the standard weak solution theory based on the integration by parts approach is no longer valid and other notions of weak solutions must be sought. Progress has been made in the latter half of the twentieth century concerning this issue, and the breakthrough occurred after the introduction of viscosity solutions by Crandall and Lions in [12]. Here, they introduced the notion of viscosity solutions and used the vanishing viscosity method to show existence of such a solution for the Hamilton–Jacobi equation:

$$u_t + H(\nabla u, u, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{1.2}$$

We recall that the vanishing viscosity method approximates the Hamilton–Jacobi equation by the following regularized second order quasilinear PDE:

$$u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + H(\nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{1.3}$$

It was shown that there exists a unique solution u^ε to (1.3) that converges locally and uniformly to a continuous function u that solves the Hamilton–Jacobi equation (1.2) in the viscosity sense [12]. However, to establish uniqueness, the following intrinsic definition of viscosity solutions was also proposed [12,13]:

Definition 1.1. Let $H : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be continuous functions, and consider the following problem:

$$H(\nabla u, u, x) = 0 \quad \text{in } \Omega, \tag{1.4a}$$

$$u = g \quad \text{on } \partial\Omega. \tag{1.4b}$$

(i) $u \in C^0(\Omega)$ is called a *viscosity subsolution* (resp., *viscosity supersolution*) of (1.4) if $u|_{\partial\Omega} = g$, and for every C^1 function $\varphi(x)$ such that $u - \varphi$ has a local maximum (resp., minimum) at $x_0 \in \Omega$, there holds $H(\nabla\varphi(x_0), u(x_0), x_0) \le 0$ (resp., $H(\nabla\varphi(x_0), u(x_0), x_0) \ge 0$).
(ii) $u \in C^0(\Omega)$ is called a *viscosity solution* of (1.4) if it is both a viscosity subsolution and supersolution.

Clearly, the above definition is not variational, as it is based on a "differentiation by parts" approach (a terminology introduced in [12,13]). In addition, the word "viscosity" loses its original meaning in the definition. However, it was shown [12,13] that every viscosity solution constructed by the vanishing viscosity method satisfies Definition 1.1. Besides addressing the uniqueness issue, another reason to favor the intrinsic differentiation by parts definition is that the definition and the notion of viscosity solutions can be readily extended to fully nonlinear second order PDEs as follows (cf. [1]):

Definition 1.2. (i) $u \in C^0(\Omega)$ is called a *viscosity subsolution* (resp., *viscosity supersolution*) of (1.1) if $u|_{\partial\Omega} = g$, and for every C^2 function $\varphi(x)$ such that $u - \varphi$ has a local maximum (resp., minimum) at $x_0 \in \Omega$, there holds $F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \le 0$. (resp., $F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \ge 0$).
(ii) $u \in C^0(\Omega)$ is called a *viscosity solution* of (1.1) if it is both a viscosity subsolution and supersolution.

For fully nonlinear *first order* PDEs, tremendous progress has been made in the past three decades in terms of PDE analysis and numerical methods. A comprehensive viscosity solution theory has been established (cf. [5,12–14]), and a wealth of efficient and robust numerical methods and algorithms have been developed and implemented (see, e.g., [15–20]). However, the situation is strikingly different for fully nonlinear *second order* PDEs. On the one hand, there have been enormous advances in PDE analysis in the past two decades (cf. [1,14]). On the other hand, the area of numerical solutions for fully nonlinear second order PDEs is still in its infancy (see the survey article [4] for recent developments). There are several reasons for this lack of progress in numerical methods. First, the most obvious difficulty is the full nonlinearity of these equations. Second, solutions to fully nonlinear second order equations are often only unique in a certain class of functions, and this conditional uniqueness is very difficult to handle numerically. Lastly and most importantly, it is extremely difficult (if all possible) to mimic the differentiation by parts approach at the discrete level. As a consequence, there is little hope to develop a discrete viscosity solution theory. Furthermore, it is not feasible to directly compute viscosity solutions using Galerkin-type numerical methods including finite element methods and spectral Galerkin methods, since they are based on variational formulations of PDEs. From a computational point of view, the notion of viscosity solutions is an "inconvenient" notion for fully nonlinear second order PDEs because it is neither constructive nor variational.

In search for a computationally better notion of weak solutions for fully nonlinear second order PDEs, we are inspired by the following simple but crucial observation: *the crux of the vanishing viscosity method for the Hamilton–Jacobi equation and the original notion of viscosity solutions is to approximate a lower order fully nonlinear PDE by a family of quasilinear higher order PDEs.* It is exactly this observation which motivated us to apply the above quoted idea to fully nonlinear second order PDE (1.1) in [21]. Namely, we approximate a "hard-to-handle" fully nonlinear second order PDE (1.1) by the following "easy-to-handle" fourth order quasilinear PDEs [21]:

$$\varepsilon\Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad \text{in } \Omega,\ \varepsilon > 0, \tag{1.5a}$$

$$u^\varepsilon = g \quad \text{on } \Omega. \tag{1.5b}$$

However, the Dirichlet boundary condition (1.5b) is not sufficient for well-posedness, and therefore an additional boundary condition must be used. Several boundary conditions could be used for this purpose, but physically, any additional boundary condition will introduce a boundary layer. Based on some heuristic arguments and evidence of numerical experiments (cf. [22]), we propose to use the following additional boundary condition:

$$\Delta u^\varepsilon = \varepsilon \quad \text{on } \partial\Omega. \tag{1.5c}$$