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An interior penalty method for distributed optimal control problems governed by the biharmonic operator

Thirupathi Gudi [a,*], Neela Nataraj [b], Kamana Porwal [a]

[a] Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

[b] Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India

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ABSTRACT

In this paper, a C^0 interior penalty method has been proposed and analyzed for distributed optimal control problems governed by the biharmonic operator. The state and adjoint variables are discretized using continuous piecewise quadratic finite elements while the control variable is discretized using piecewise constant approximations. *A priori* and *a posteriori* error estimates are derived for the state, adjoint and control variables under minimal regularity assumptions. Numerical results justify the theoretical results obtained. The *a posteriori* error estimators are useful in adaptive finite element approximation and the numerical results indicate that the sharp error estimators work efficiently in guiding the mesh refinement.

1. Introduction

Consider the distributed optimal control problem defined by

$$\min_{u\in U_{ad}} J(y,u) \text{ subject to} \tag{1.1a}$$

$$\Delta^2 y = f + u \quad \text{in } \Omega, \tag{1.1b}$$

$$y|_{\partial\Omega} = \left.\frac{\partial y}{\partial n}\right|_{\partial\Omega} = 0, \tag{1.1c}$$

where the domain $\Omega \subset \mathbb{R}^2$ is assumed to be bounded polygonal with boundary $\partial\Omega$, the load function $f \in H^{-1}(\Omega)$, $U_{ad} \subset L^2(\Omega)$ is a non-empty, convex and bounded admissible set of controls defined by

$$U_{ad} = \{u \in L^2(\Omega) : u_a \le u(x) \le u_b \text{ a.e. in } \Omega\}, \tag{1.2}$$

$u_a, u_b \in \mathbb{R} \cup \{\pm\infty\},\ u_a < u_b$ are given and the cost functional $J(y,u)$ is defined by

$$J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \tag{1.3}$$

with a fixed regularization parameter $\alpha > 0$ and y_d is the given observation for y.

This paper discusses a C^0 interior penalty (C^0 IP) method based discretization of (1.1a)–(1.1c) and develops *a priori* and *a posteriori* error estimates for the state, adjoint and control variables in polygonal domains with possible corner singularities.

* Corresponding author.
E-mail addresses: gudi@math.iisc.ernet.in (T. Gudi), neela@math.iitb.ac.in (N. Nataraj), kamana@math.iisc.ernet.in (K. Porwal).

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Within the finite element framework, the fourth order problems can be solved using conforming, non-conforming or mixed finite element methods. C^0 IP methods have become attractive alternatives to solve fourth order problems in the recent past due to the fact that they use standard C^0 Lagrange finite element spaces which are designed for second order problems and the derivation of variational forms is flexible in terms of handling even complicated problems and also yield quasi-uniform estimates [1–6]. Discontinuous Galerkin methods [7–9], to cite a few references have also become attractive for fourth order problems in different aspects although they involve larger numbers of degrees of freedom than that of C^0 IP methods.

Though the approximation theory of discretized optimal control problems for second order problems is well-developed, except for [10–12], not much literature is available for the finite element analysis and *a priori* /*a posteriori* estimates of optimal control problems governed by fourth order elliptic equations. *A priori* error estimates for the approximate state and control for a distributed optimal control problem governed by the biharmonic equation with discretization based on the mixed formulation designed by Hellan–Herrmann–Miyoshi defined on convex polygons have been discussed in [11]. The analysis in this work heavily relies on the fact that the domain is a *convex polygon* and this guarantees the assumed *additional regularity* of the exact solution. But in general, the regularity of solution for a fourth order problem in polygonal domains is limited. For instance, for the biharmonic equation defined in a polygonal domain with Dirichlet boundary conditions, when $f \in H^{-1}(\Omega)$ and $u \in L^2(\Omega)$, the solution of the state equation $y \in H^{2+\gamma}$, where the *elliptic regularity index* $\gamma \in (1/2, 1]$, with $\gamma = 1$, if Ω is convex and $\gamma < 1$ if Ω is non-convex. Since the standard techniques are no longer applicable when extra regularity assumptions on the exact solution is not available, in this paper, we combine ideas used in *a posteriori* analysis and *a priori* analysis, that is, we use a medius analysis [13] to establish both *a priori* and *a posteriori* estimates for optimal control problems governed by the biharmonic operator. That is, under realistic regularity assumptions for the problem defined on a *polygonal domain*, we

- establish *a priori* error estimates for the state and adjoint variables in the energy, H^1 and L^2 norms, an L^2 convergence estimate for the control variable and a super convergence result for the post-processed control in convex domains;
- derive reliable and efficient residual based *a posteriori* error estimators for the state, adjoint and control variables which drive the adaptive mesh refinements;
- perform numerical experiments that substantiate the theoretical results.

To our knowledge, this is the first attempt to study the discretization errors for optimal control problems governed by higher order equations under lower regularity assumptions using both *a priori* and *a posteriori* approaches.

The rest of the paper is organized as follows. We describe the weak formulation for the optimal control problem in Section 2. Section 3 deals with the C^0 IP formulation. *A priori* error estimates for the state, adjoint and control variables followed by numerical results that justify the estimates are presented in Section 4. Reliable and efficient *a posteriori* error estimators for the state, adjoint and control variables are established in Section 5 and this is followed by numerical examples that illustrate the performance of the error estimator.

2. Weak formulation

Recasting (1.1a)–(1.1c) in the weak form yields

$$\min_{(y,u)\in V\times U_{ad}} J(y,u) \text{ subject to} \tag{2.1a}$$

$$a(y,\phi) = (f+u,\phi) \quad \forall \phi \in V, \tag{2.1b}$$

where $V = H_0^2(\Omega)$, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bilinear form defined by $a(\phi, w) = \int_\Omega D^2\phi : D^2 w dx \quad \forall \phi, w \in V$ with $D^2\phi : D^2 w = \sum_{i,j=1}^{2} \frac{\partial^2\phi}{\partial x_i \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_j}$, satisfying V ellipticity, that is,

$$\exists C > 0 \text{ such that } a(\phi,\phi) \geq C\|\phi\|_V^2. \tag{2.2}$$

For notational convenience, we denote both the L^2 inner product and the duality pairing between $f \in H^{-1}(\Omega)$ and $\phi \in V \subset H_0^1(\Omega)$ defined on Ω using $(\cdot,\cdot)$ and distinguishing them as inner product or duality pairing is easily clear from the context. The L^2 norm defined on Ω is denoted as $\|\cdot\|$. For any measurable set $T \subset \Omega$, the notations $(\cdot,\cdot)_{0,T}$ and $\|\cdot\|_{0,T}$ are used to denote the L^2 inner product and norm on T. Standard notions of function spaces and norms are used throughout in the paper, unless mentioned otherwise.

It is well-known [14,15] that the convex control problem (2.1a)–(2.1b) has a unique solution $(\bar{y}, \bar{u}) \in V \times U_{ad}$ and there exists a co-state $\bar{p} \in V$ such that the triplet $(\bar{y}, \bar{p}, \bar{u})$ satisfies the Karush–Kuhn–Tucker (KKT) optimality conditions [14]:

$$a(\bar{y},\phi) = (f+\bar{u},\phi) \quad \forall \phi \in V, \tag{2.3a}$$

$$a(\bar{p},\phi) = (\bar{y}-y_d,\phi) \quad \forall \phi \in V, \tag{2.3b}$$

$$(\alpha\bar{u}+\bar{p}, v-\bar{u}) \geq 0 \quad \forall v \in U_{ad}. \tag{2.3c}$$

The optimal control $\bar{u}$ in (2.3c) has the representation a.e. for $x \in \Omega$:

$$\bar{u}(x) = \pi_{[u_a,u_b]}\left(-\frac{1}{\alpha}\bar{p}(x)\right), \tag{2.4}$$