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A hybridized finite element method for the Stokes problem

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ABSTRACT

A locally conservative hybridized finite element method for Stokes equations is presented and analyzed. The hybridized approach reduces a lot of degrees of freedom, especially for pressure approximation. In our approach the pressure is determined locally up to a constant, therefore, the global stiffness system contains only the average of pressure variable on each cell as unknowns.

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1. Introduction

Let us consider the Stokes equations:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{u} = 0 \quad \text{on } \Gamma, \tag{1.1c}$$
$$(p, 1)_{\Omega} = 0, \tag{1.1d}$$

where the domain Ω is a simply connected, bounded, convex polygonal domain. The variables $(\mathbf{u}, p)$ represent the velocity vector field and pressure of a flow, respectively, where the external force $\mathbf{f} \in [L^2(\Omega)]^2$ applies.

The hybridized finite element method has been successfully applied to elliptic equations, elasticity equations and Stokes equations [1–4]. Advantages of the hybridized approach are that it has the degrees of freedom only on the skeleton of a triangulation (natural static condensation property) and it satisfies local conservation properties. For elasticity problems a mixed formulation can be used to avoid locking phenomena by introducing an artificial pressure. The hybridized formulation induces a stiffness system only with the skeleton data of the primal variable as unknown. For flow problems such as Stokes and Navier–Stokes equations reduction in the degrees of freedom is more phenomenal. In this case, the global unknown becomes the skeleton data of the velocity profile and the cell average of the pressure since the pressure will be determined up to constant during the local static condensation process.

The hybridized approach is composed of two steps. In Step 1, localize the problem on each cell and decompose the local equation into two parts: The homogeneous equation which assumes the skeleton values of the solution on the each cell boundary as the Dirichlet data; and the nonhomogeneous equation corresponding to the source $\mathbf{f}$ with the null Dirichlet data on each cell boundary. In Step 2, set up the global stiffness system by using the normal traction continuity on intercell interfaces. Hence, the unknowns are the assumed values of the solution on the skeleton of a triangulation and the average

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of pressure on each cell since the pressure can be determined up to a constant on each cell by the localization process. For numerical solvers in Step 1 there can be many different approaches; a LDG (local discontinuous Galerkin) type solver [5,4], a penalty-type local solver [6] or a Baumann–Oden type local solver [1,2]. Step 1 can be understood as a process of constructing PDE-adapted basis functions as shown in the multiscale finite element methods (MsFEM) [7]. However, our local solver is of a spectral type in contrast to the fine scale finite element in the MsFEM. The local conservation in each cell will be guaranteed by Step 1 and the weak continuity of total traction on each cell interface is provided by Step 2. Therefore, the characteristic of the hybridized approach is very similar to that of the finite volume type methods.

In this paper we consider a hybridized numerical scheme based on the Baumann–Oden type local solver. As in the theory of finite element methods our method is not stable for the P_1–P_0 and Q_1–Q_0 pairs [8,9]. Hence, our analysis covers the case for Q_k–Q_{k-1} with $k \geq 2$. A notable feature of our approach is that it suffices to show the inf–sup condition for Q_k–Q_0 pair for complete stability and error analysis, which can be easily proved. It is worth mentioning that those analysis for rectangular elements can be easily extended to cover the triangular case.

The aim of our approach is primarily to design a numerical method that satisfy the local flux conservation without introducing any dual partition or dual variable as in the finite volume method or the mixed method, respectively. The natural static condensation property embedded is an additional advantage of our approach.

The paper is organized as follows. In Section 2 we introduce function spaces and the hybridization technique. The finite element approach based on the Baumann–Oden local solver is introduced in Section 3. In addition some norm equivalence between the skeleton function space and solution space is discussed. Section 4 is devoted for numerical analysis. In it we introduce an interpolation and a local solution projection with a conservation property, which will play essential roles for the proof of the main theorem. In the last section we provide a comparison of computational costs between the hybridized method and the classical finite element method. We discuss the Uzawa algorithm for the pressure update and extension of our analysis to the case of triangular elements.

2. Hybridization

Let $(\mathcal{T}_h)_{0<h\leq 1}$ be a family of shape regular, quasi uniform triangular or rectangular subdivision of Ω with $\max_{T\in\mathcal{T}_h} \operatorname{diam}(T) = h$. Denote by E_h the set of all edges in $\mathcal{T}_h$ and by K_h the *skeleton* $\cup_{e\in E_h} e$ of triangulation $\mathcal{T}_h$.

For a set $D \subset \mathbb{R}^2$ ($D = T$ or $D = \Omega$ is possible), $H^s(D)$ denotes the usual Sobolev space with $s \geq 0$, of which norms and seminorms are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. The broken Sobolev space $H^s(\mathcal{T}_h) = \Pi_{T\in\mathcal{T}_h} H^s(T)$ is equipped with the norm, $\|u\|_{s,h} := (\sum_{T\in\mathcal{T}_h} \|u\|_{s,T}^2)^{1/2}$. Let $\widetilde{H}^s(\mathcal{T}_h) = \Pi_{T\in\mathcal{T}_h}\widetilde{H}^s(T)$ with $\widetilde{H}^s(T) = \{v \in H^s(T) | \int_T v\mathrm{d}x = 0\}$. The space $H_0^s(\mathcal{T}_h)$ denotes the subspace $\{v \in H^s(\mathcal{T}_h) : v \text{ vanishes on } \partial T \cap \partial(\Omega)\}$for $s \geq 1/2$. Denote by $(\cdot,\cdot)_{\mathcal{S}}$ and $\langle\cdot,\cdot\rangle_{\partial\mathcal{S}}$ the L^2-inner products on $\mathcal{S}$ and $\partial\mathcal{S}$, respectively. Our numerical analysis is mostly based on the discrete inner product $(\nabla_h u, \nabla_h v)_\Omega := \sum_{T\in\mathcal{T}_h}(\nabla u, \nabla u)_T$.

We now introduce some function spaces on the skeleton K_h:

$$H^{1/2}(K_h) = \{u|_{K_h} : u \in H^1(\Omega)\}; \qquad H_0^{1/2}(K_h) = \{u|_{K_h} : u \in H_0^1(\Omega)\},$$

with norm

$$\|\phi\|_{1/2,K_h}^2 = \sum_{T\in\mathcal{T}_h} \|\phi\|_{1/2,\partial T}^2, \tag{2.1}$$

where the trace norm is given as $\|\phi\|_{1/2,\partial T}^2 = \inf_{u\in H^1(T), u|_{\partial T}=\phi} \|u\|_{1,T}^2$.

For a pair of velocity field and pressure $(\mathbf{u}, p)$ denote the normal traction on the boundary of K by

$$\mathcal{D}^{\boldsymbol{\nu}}(\mathbf{u}, p) = \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} - p\boldsymbol{\nu}, \tag{2.2}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal. Then, with the introduction of an auxiliary trace variable $\boldsymbol{\phi} \in [H_0^{1/2}(K_h)]^2$ on the skeleton K_h, the first step of hybridization [8] is to decompose the problem (1.1) into the following localized coupled system: for $T \in \mathcal{T}_h$

$$-\Delta\mathbf{u} + \nabla\widetilde{p} = \mathbf{f} \quad \text{in } T, \tag{2.3a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } T, \tag{2.3b}$$

$$\mathbf{u} = \boldsymbol{\phi} \quad \text{on } \partial T, \tag{2.3c}$$

$$(\widetilde{p}, 1)_T = 0 \tag{2.3d}$$

and the continuity of the traction across all the intercell boundaries and the divergence free condition:

$$[\![\mathcal{D}^{\boldsymbol{\nu}}(\mathbf{u}, \widetilde{p} + p_0)]\!]_{K_h} = 0 \quad \text{and} \quad \int_{\partial T} \boldsymbol{\phi}\cdot\boldsymbol{\nu} ds = 0, \tag{2.4}$$

where $[\![\phi]\!]_{K_h}$ denotes the jump of ϕ across the skeleton K_h and $p_0 = \frac{1}{|T|}\int_T p\mathrm{d}x$.

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