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A multiscale discontinuous Galerkin method for convection–diffusion–reaction problems☆

Mi-Young Kim [a,*], Mary F. Wheeler [b]

[a] Department of Mathematics, Inha University, Incheon 402-751, Republic of Korea

[b] Institute for Computational Engineering and Sciences, Center for Subsurface Modelling, The University of Texas at Austin, Austin, TX, 78712-0027, USA

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ABSTRACT

We provide a general framework of multiscale discontinuous Galerkin methods developed in Buffa et al. (2006), Hughes et al. (2006) for general second-order partial differential equations. We establish stability of the method and prove the error estimates.

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1. Introduction

Over the past decades, there has been much development in discontinuous Galerkin (DG) methods. Cockburn et al. have worked extensively on the development and analysis of DG methods for variety of nonlinear hyperbolic conservation laws [1]. Examples of the DG methods include the Bassi–Rebay method [2], the Local Discontinuous Galerkin (LDG) methods [3,4], the Oden–Babuška–Baumann (OBB-DG) method [5], and interior penalty Galerkin methods [6–8]. For DG methods applied to general problems, we also refer to [1,9,10], and the references cited therein. Compared with continuous Galerkin methods, the DG methods possess better properties in many situations but they require much large number of degrees-of-freedom, [11,12].

Recently, there have been shown much efficient methods related with DG method. For instance, multiscale mortar DG–DG coupling methods were introduced in [13,14] based on four different DG formulations, the OBB-DG [5], the nonsymmetric interior penalty Galerkin (NIPG) [15], the symmetric interior penalty Galerkin (SIPG) [8,16–18], and the incomplete interior penalty Galerkin (IIPG) [6,17,18]. There, the mortar variable was used as a Lagrange multiplier on the interface. The computational problem reduces to the interface problem having reduced size of the system. Hybridizable DG methods were also introduced in [19,20] to reduce the number of degrees-of-freedom of DG. On the other hand, a multiscale discontinuous Galerkin (MDG) method was developed in [21,22] for advection–diffusion problems. The solution of the MDG method is expressed as a sum of the fine scale ("local") and coarse scale components. The solutions of the "local" problems are discontinuous but they are parameterized by the degrees-of-freedom of continuous space (coarse scale). The discrete problem of the MDG method has the equation size of the one of the continuous Galerkin method. The MDG method retains the quality of the (donor) DG method and has potential of computational cost of continuous Galerkin methods. The authors of [21,22] numerically studied the MDG method and showed that the inf–sup constant is positive, bounded uniformly from zero. Mathematical analysis was not treated there. In this paper, we provide analysis for the stability and the error estimates

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* Corresponding author. Tel.: +82 32 860 7633; fax: +82 32 874 5615.
E-mail addresses: mikim@inha.ac.kr (M.-Y. Kim), mfw@ices.utexas.edu (M.F. Wheeler).

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of the MDG method. In [21], a stabilized variant of the MDG method (SMDG) was also introduced. The authors modified the (donor) DG method by adding an SUPG stabilization term and obtained the stabilized DG (SDG) method. They then defined the SMDG method with the (donor) SDG method. They obtained the same error estimates as for the (donor) SDG method.

In this paper, we provide a general framework of the MDG method and establish mathematical analysis for general second-order partial differential equations. We assume heterogeneous anisotropic diffusion tensor which is allowed to be degenerate. We formulate the MDG method systematically by defining the local problem in such a way that the (global) MDG method is consistent. We show the stability of the method and prove the error estimates. For the convergence analysis, we first establish the approximation properties of the approximation space which consists of the solution spaces of the local problems. Using the local properties of the method, we then prove the error estimates for the solution to the MDG method.

The organization of the remainder of this paper is as follows. In Section 2, we introduce the model problem. In Section 3, we present the finite element space for the MDG method. In Section 4, we provide the general framework of the MDG method. Finally, in Section 5, we apply the MDG method to the model problem. We show stability of the local and global problems. We prove the error estimates of the approximate solution.

2. Model problem

Let Ω be a bounded open domain in $\mathbb{R}^m$, $m = 2, 3$ with a polyhedral boundary $\partial\Omega$. We consider the advection–diffusion–reaction equation

$$-\nabla\cdot(\mathbf{a}(x)\nabla u) + \nabla\cdot(\boldsymbol{\beta}(x)u) + c(x)u = f(x), \tag{2.1}$$

where $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$ are real-valued, $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^m$ is a vector function whose entries β_i are Lipschitz continuous real valued function on $\overline{\Omega}$, and $\mathbf{a} = (a_{ij})_{i,j=1}^m$ is a symmetric matrix whose entries are bounded, piecewise continuous real-valued functions defined on $\overline{\Omega}$, with

$$\boldsymbol{\zeta}^T\mathbf{a}(x)\boldsymbol{\zeta} \geq 0 \quad \forall\boldsymbol{\zeta}\in\mathbb{R}^m, \text{ a.e. } x\in\overline{\Omega}. \tag{2.2}$$

By the square root lemma, the matrix function $\mathbf{a}$ then admits a unique (symmetric) square root $\sqrt{\mathbf{a}}$ and it satisfies

$$\mathbf{a}\mathbf{w}\cdot\mathbf{v} = \sqrt{\mathbf{a}}\mathbf{w}\cdot\sqrt{\mathbf{a}}\mathbf{v} \quad \forall\mathbf{w},\ \mathbf{v}\in\mathbb{R}^m. \tag{2.3}$$

We also adopt the following hypothesis: There exists a positive function c_o such that

$$(c_o(x))^2 = c(x) + \frac{1}{2}\nabla\cdot\boldsymbol{\beta}(x) \quad \text{a.e. } x\in\Omega. \tag{2.4}$$

By $\mathbf{n}(x)$ we denote the unit outward normal vector to $\partial\Omega$ at $x\in\partial\Omega$. We define

$$\partial\Omega_- = \{x\in\partial\Omega : \boldsymbol{\beta}(x)\cdot\mathbf{n}(x) < 0\} \quad \text{and} \quad \partial\Omega_+ = \{x\in\partial\Omega : \boldsymbol{\beta}(x)\cdot\mathbf{n}(x)\geq 0\}.$$

The sets $\partial\Omega_-$ and $\partial\Omega_+$ will be referred to as the *inflow* and *outflow* boundaries, respectively. We supplement (2.1) with the boundary conditions

$$(\boldsymbol{\beta}u + \mathbf{a}\nabla u)\cdot\mathbf{n} = g_I\boldsymbol{\beta}\cdot\mathbf{n} \quad \text{on } \partial\Omega_-, \qquad \mathbf{a}\nabla u\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega_+. \tag{2.5}$$

3. Finite element spaces

In this section, we introduce the finite element spaces for the MDG method. Let $\mathcal{T}$ be a regular family of triangulations of Ω in the sense that there exists a $\kappa > 0$ such that $\frac{h}{h_{\min}} < \kappa$, where $h_{\min} = \min_{K\in\mathcal{T}} h_K$, h_K is the diameter of $K\in\mathcal{T}$, and $h = \max_{K\in\mathcal{T}} h_K$. We assume that $\mathcal{T}$ contains only regular nodes, that is, each element vertex is also a vertex to all adjacent elements and there are no hanging nodes. The elements $K\in\mathcal{T}$ are either triangles and/or quadrilaterals in two dimensions or tetrahedra and/or hexahedra in three dimensions. We denote by $\mathcal{E}$ the set of all edges of $\mathcal{T}$, by $\mathcal{E}_o$ the set of all interior edges, and by $\mathcal{E}_\partial = \mathcal{E}\setminus\mathcal{E}_o$ the set of all boundary edges.

In consistent DG methods, the solution values are coupled by generalized flux functions across the edges and they appear by the *jumps* and *averages*. We define a partition of the elementary boundary ∂K for $K\in\mathcal{T}$ as follows:

$$\partial_- K = \{x\in\partial K : \boldsymbol{\beta}(x)\cdot\mathbf{n}(x) < 0\}, \qquad \partial_+ K = \{x\in\partial K : \boldsymbol{\beta}(x)\cdot\mathbf{n}(x)\geq 0\}, \tag{3.1}$$

where $\mathbf{n} = \mathbf{n}_K$ is used, with abuse of notation, as the outward unit normal with respect to ∂K and $\partial_\mp K$ represent the element *inflow*/*outflow* boundaries, respectively. For an interior edge or face $e\in\mathcal{E}_o$, if $\boldsymbol{\beta}\not\equiv 0$ on e, we choose the normal $\mathbf{n}_e$ for which $\mathbf{n}_e\cdot\boldsymbol{\beta}\geq 0$. Since e is shared by exactly two elements, the outward normal on one of these element will coincide with $\mathbf{n}_e$; we call this element the upwind element and denote by K^-. The outward normal on the other element will have the opposite direction to the normal on e; we call this element downwind element and denote by K^+. We denote by $\mathbf{n}^-$ and $\mathbf{n}^+$ their respective outward-pointing unit normals. The *jumps* and *averages* are then defined on e with $\varphi^\pm = \varphi_{|_{K^\pm}}$ by

$$\{\varphi\} = \frac{1}{2}(\varphi^- + \varphi^+), \qquad [|\varphi|] = \varphi^-\mathbf{n}^- + \varphi^+\mathbf{n}^+. \tag{3.2}$$

For a vector-valued function $\boldsymbol{\tau}$, element-wise smooth function, with analogous meaning for $\boldsymbol{\tau}^-$ and $\boldsymbol{\tau}^+$, we define

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^- + \boldsymbol{\tau}^+), \qquad [|\boldsymbol{\tau}|] = \boldsymbol{\tau}^-\cdot\mathbf{n}^- + \boldsymbol{\tau}^+\cdot\mathbf{n}^+.$$

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