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An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes

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ABSTRACT

This paper presents a new and efficient numerical algorithm for the biharmonic equation by using weak Galerkin (WG) finite element methods. The WG finite element scheme is based on a variational form of the biharmonic equation that is equivalent to the usual H^2-semi norm. Weak partial derivatives and their approximations, called discrete weak partial derivatives, are introduced for a class of discontinuous functions defined on a finite element partition of the domain consisting of general polygons or polyhedra. The discrete weak partial derivatives serve as building blocks for the WG finite element method. The resulting matrix from the WG method is symmetric, positive definite, and parameter free. An error estimate of optimal order is derived in an H^2-equivalent norm for the WG finite element solutions. Error estimates in the usual L^2 norm are established, yielding optimal order of convergence for all the WG finite element algorithms except the one corresponding to the lowest order (i.e., piecewise quadratic elements). Some numerical experiments are presented to illustrate the efficiency and accuracy of the numerical scheme.

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1. Introduction

This paper is concerned with new developments of numerical methods for the biharmonic equation with Dirichlet and Neumann boundary conditions. The model problem seeks an unknown function $u = u(x)$ satisfying

$$\begin{aligned} \Delta^2 u &= f, \quad \text{in } \Omega, \\ u &= \xi, \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \nu, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz continuous boundary $\partial\Omega$. The functions f, ξ, and ν are given on the domain or its boundary, as appropriate.

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A variational formulation for the biharmonic problem (1.1) is given by seeking $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = \xi$, $\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \nu$ and the following equation

$$\sum_{i,j=1}^{d} (\partial_{ij}^2 u, \partial_{ij}^2 v) = (f, v), \quad \forall v \in H_0^2(\Omega), \tag{1.2}$$

where $(\cdot, \cdot)$ stands for the usual inner product in $L^2(\Omega)$, ∂_{ij}^2 is the second order partial derivative in the directions x_i and x_j, and $H_0^2(\Omega)$ is the subspace of the Sobolev space $H^2(\Omega)$ consisting of functions with vanishing trace for the function itself and its gradient.

Based on the variational form (1.2), one may design various conforming finite element schemes for (1.1) by constructing finite element spaces as subspaces of $H^2(\Omega)$. Such H^2-conforming methods essentially require C^1-continuity for the underlying piecewise polynomials (known as finite element functions) on a prescribed finite element partition. The C^1-continuity imposes an enormous difficulty in the construction of the corresponding finite element functions in practical computation. Due to the complexity in the construction of C^1-continuous elements, H^2-conforming finite element methods are rarely used in practice for solving the biharmonic equation.

As an alternative approach, nonconforming and discontinuous Galerkin finite element methods have been developed for solving the biharmonic equation over the last several decades. The Morley element [1] is a well-known example of the nonconforming element for the biharmonic equation by using piecewise quadratic polynomials. Recently, a C^0 interior penalty method was studied in [2,3]. In [4], a hp-version interior-penalty discontinuous Galerkin method was developed for the biharmonic equation. To avoid the use of C^1-elements, mixed methods have been developed for the biharmonic equation by reducing the fourth order problem to a system of two second order equations [5–9].

Recently, weak Galerkin (WG) has emerged as a new finite element technique for solving partial differential equations. The WG method refers to numerical techniques for partial differential equations where differential operators are interpreted and approximated as distributions over a set of generalized functions. The method/idea was first introduced in [10] for second order elliptic equations, and the concept was further developed in [11–13]. By design, WG uses generalized and/or discontinuous approximating functions on general meshes to overcome the barrier in the construction of "smooth" finite element functions. In [14], a WG finite element method was introduced and analyzed for the biharmonic equation by using polynomials of degree $k \geq 2$ on each element plus polynomials of degree k and $k-1$ for u and $\frac{\partial u}{\partial \mathbf{n}}$ on the boundary of each element (i.e., elements of type $P_k/P_k/P_{k-1}$). The WG scheme of [14] is based on the variational form of $(\Delta u, \Delta v) = (f, v)$.

In this paper, we will develop a highly flexible and robust WG finite element method for the biharmonic equation by using an element of type $P_k/P_{k-2}/P_{k-2}$; i.e., polynomials of degree k on each element and polynomials of degree $k-2$ on the boundary of the element for u and ∇u. Our WG finite element scheme is based on the variational form (1.2), and has a smaller number of unknowns than that of [14] for the same order of element. Intuitively, our WG finite element scheme for (1.1) shall be derived by replacing the differential operator ∂_{ij}^2 in (1.2) by a discrete and weak version, denoted by $\partial_{ij,w}^2$. In general, such a straightforward replacement may not produce a working algorithm without including a mechanism that enforces a certain weak continuity of the underlying approximating functions. A weak continuity shall be realized by introducing an appropriately defined stabilizer, denoted as $s(\cdot, \cdot)$. Formally, our WG finite element method for (1.1) can be described by seeking a finite element function u_h satisfying

$$\sum_{i,j=1}^{d} (\partial_{ij,w}^2 u_h, \partial_{ij,w}^2 v)_h + s(u_h, v) = (f, v) \tag{1.3}$$

for all testing functions v. The main advantage of the present approach as compared to [14] lies in the fact that elements of type $P_k/P_{k-2}/P_{k-2}$ are employed, which greatly reduces the degrees of freedom and results in a smaller system to solve. The rest of the paper is to specify all the details for (1.3), and justifies the rigorousness of the method by establishing a mathematical convergence theory.

The paper is organized as follows. In Section 2, we introduce some standard notations for Sobolev spaces. Section 3 is devoted to a discussion of weak partial derivatives and their discretizations. In Section 4, we present a weak Galerkin algorithm for the biharmonic equation (1.1). In Section 5, we introduce some local L^2 projection operators and then derive some approximation properties which are useful in the convergence analysis. Section 6 will be devoted to the derivation of an error equation for the WG finite element solution. In Section 7, we establish an optimal order of error estimate for the WG finite element approximation in a H^2-equivalent discrete norm. In Section 8, we shall derive an error estimate for the WG finite element method approximation in the usual L^2-norm. Finally in Section 9, we present some numerical results to demonstrate the efficiency and accuracy of our WG method.

2. Preliminaries and notations

Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$, $d = 2, 3$. We use the standard definition for the Sobolev space $H^s(D)$ and the associated inner product $(\cdot, \cdot)_{s,D}$, norm $\|\cdot\|_{s,D}$, and seminorm $|\cdot|_{s,D}$ for any $s \geq 0$. For