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A nonconforming scheme with high accuracy for the plate bending problem

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ABSTRACT

The convergence order $O(h^2)$ of the plate bending problem has been derived by the superconvergence methods or constructing new finite elements where shape function space consist of P_3 space so far. In this paper, we can also present the same result of the Adini rectangular element by the interior penalty method. We built a new nonconforming scheme, where the penalty parameter is accurately estimated and the consistency term vanishes. Then, the error estimate can only be determined by the interpolation error.

1. Introduction

Consider the plate bending problem: find $u \in H_0^2(\Omega)$, such that

$$a(u, v) = (f, v), \quad \text{for all } v \in H_0^2(\Omega), \tag{1}$$

where

$$a(u, v) = \int_{\Omega} \Delta u \Delta v + (1 - r)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)dxdy, \tag{2}$$

and $r \in (0, \frac{1}{2}]$ is the Poisson ratio of the plate. It requires the continuity of the first-order partial derivatives across adjacent finite elements when using conforming finite element methods approximation of this problem, which leads to too much degrees of freedom. To relax this constraint, many kinds of nonconforming finite elements have been constructed, such as the Morley element [1], the NZT element [2] and Veubake element [3]. All these nonconforming elements are convergent with order $O(h)$. On how to improve the convergent accuracy, a large amount of work can be found.

Ciarlet [4] derived a convergence rate $O(h)$ approximation of the biharmonic equation for the Adini rectangular element. Its interpolation error is of order $O(h^2)$, which is one order higher than the consistent error; Wang [5] gave the nearly optimal L^∞ estimates of the Adini element for the biharmonic equation as follows:

When $u \in H_0^2(\Omega) \cap H^3(\Omega)$,

$$|u - u_h|_{0,\infty,\Omega} \le Ch^2 |\ln h|^{\frac{1}{2}} |u|_{3,\Omega}.$$

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When $u \in W^{3,\infty}(\Omega)$,

$$|u - u_h|_{1,\infty,\Omega} \le Ch^2 |\ln h|^{\frac{5}{4}} |u|_{3,\infty,\Omega}.$$

Lin and Yan [6] got the superclose result of the Adini element on uniform rectangular meshes; Furthermore, the superconvergence of Adini element for the biharmonic equation on anisotropic meshes was obtained in [7]. In [8], the superconvergent result of the triangular Morley plate element had been obtained as

$$|u - I_{3h}^3 u_h| \le Ch^{\frac{3}{2}} (\|u\|_{4,\Omega} + \|f\|_{0,\Omega})$$

by the post processing interpolation technique. Gao et al. [9] presented a new non-conforming plate element with a convergence rate of $O(h^2)$.

In this paper, we also present a convergence rate of $O(h^2)$ by the interior penalty method for the plate bending problem. The study of interior penalty method tracks back to the 1970s. For fourth order problem, Baker [10] used this method to impose C^1 interelement continuity on C^0 elements and proved the optimal convergence in L^2. Engel et al. [11] proposed an interior penalty method that uses only the standard C^0 finite elements for second order problems. Brenner and Sung [12] analyzed the C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains using the Lagrange finite element. A higher convergence order is proved, but the penalty parameter is not accurate. Brenner et al. [13] derived a posteriori error estimator for the biharmonic problem by the interior penalty method. Brenner et al. [14] developed isoparametric C^0 interior penalty methods for plate bending problems on smooth domains and proved the optimal convergence in the energy norm. In view of these results, when using the Adini nonconforming element approximation of the plate bending problem, we built a new discrete weak form by the interior penalty method, where the penalty parameter is accurately estimated and the consistency term vanishes. Therefore, the error estimate can only be determined by the interpolation error.

The rest of the paper is organized as follows. We introduce the approximation of the plate bending problem along with some notations and preliminaries in Section 2. The accurate estimation of the penalty parameter is given in Section 3. Section 4 contains the convergent accuracy of the new scheme. Finally, a numerical experiment is carried out in Section 5.

2. The new nonconforming scheme

Let $\hat{T} = [-1, 1] \times [-1, 1]$ be a reference element with vertices $\hat{A}_1 = (-1, -1), \hat{A}_2 = (1, -1), \hat{A}_3 = (1, 1), \hat{A}_4 = (-1, 1)$. Let $\hat{l}_1 = \overline{\hat{A}_1\hat{A}_2}, \hat{l}_2 = \overline{\hat{A}_2\hat{A}_3}, \hat{l}_3 = \overline{\hat{A}_3\hat{A}_4}, \hat{l}_4 = \overline{\hat{A}_4\hat{A}_1}$ be the four sides of $\hat{T}$.

The ACM finite element $(\hat{T}, \hat{P}, \hat{\Sigma})$ is defined as

$$\hat{P} = P_3(\hat{T}) \oplus span\{\hat{x}^3\hat{y}, \hat{x}\hat{y}^3\}, \qquad \hat{\Sigma} = \{\hat{v}(\hat{A}_i), \hat{v}_{\hat{x}}(\hat{A}_i), \hat{v}_{\hat{y}}(\hat{A}_i), 1 \le i \le 4\}.$$

$\hat{\Pi} : H^3(\hat{T}) \to \hat{P}$ is the interpolate operator on $\hat{T}$.

Let Ω be a rectangular domain with boundaries $\partial\Omega$ parallel to the coordinate axes. $\{\mathcal{T}_h\}$ is a rectangular subdivision of Ω with the regular assumption. Let $T \in \mathcal{T}_h$ be a rectangle, with the central point (x_T, y_T), $2h_x$ and $2h_y$ the length of sides parallel to x axis and y axis respectively, $A_1(x_T - h_x, y_T - h_y)$, $A_2(x_T + h_x, y_T - h_y)$, $A_3(x_T + h_x, y_T + h_y)$ and $A_4(x_T - h_x, y_T + h_y)$ the four vertices. $h_T = \max\{h_x, h_y\}$, $h = \max_{T \in \mathcal{T}_h} h_T$, $\rho_T = \sup\{\text{diam} S : S \subset T \text{ is a circle}\}$. Then the regular assumption is $\frac{h_T}{\rho_T} \le \sigma$, $\forall T \in \mathcal{T}_h$ (the σ is a positive constant independent of $\mathcal{T}_h$ and of the function under consideration).

The affine mapping $F_T : \hat{T} \to T$ is defined as

$$\begin{cases} x = h_x\hat{x} + x_T, \\ y = h_y\hat{y} + y_T. \end{cases}$$

Define the finite element space V_h as

$$V_h = \left\{ v_h : v_h|_T = \hat{v}_h \circ F_T^{-1}, \hat{v}_h \in \hat{P}, \forall T \in \mathcal{T}_h; v_h(a) = \frac{\partial v_h}{\partial n}(a) = 0 \text{ for all nodes } a \text{ on } \partial\Omega \right\}.$$

The interpolation operator Π_h is defined as

$$\Pi_h : H^3(\Omega) \to V_h,\ \Pi_h|_T = \Pi_T,\ \Pi_T v = \hat{\Pi}\hat{v} \circ F_T^{-1}.$$

We introduce the jump and average of a scalar function f as follows. Let E be an edge shared by two elements T and T'. Then the jump of f over E is defined by

$$[f] = f|_T - f|_{T'}$$

and the average as

$$\{f\} = \frac{1}{2}(f|_T + f|_{T'}).$$