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Sufficient condition of stability of finite element method for symmetric T-hyperbolic systems with constant coefficients

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ABSTRACT

In this note, Finite Element Method is applied to solve the symmetric t-hyperbolic system with dissipative boundary condition and its stability is proved. In two-dimensional space, complex program is developed for the numerical solution of the mixed problem in simple connected region on the uniform grid. Delphi-7 is used for the code of the complex program. Numerical results are in line with the theoretical findings.

0. Introduction

There are many results [1–10] that are devoted to the study of finite element method (FEM) for a mixed problem for symmetric hyperbolic systems with the help of finite differences method. Various explicit and implicit finite difference schemes (FDS) are obtained. The stability of the obtained schemes is also investigated. Error estimates of the approximate solutions of the mixed problems are given.

In [5], the explicit (in time) difference scheme is constructed based on the FEM for a mixed problem of symmetric t-hyperbolic systems (see also [1]) in the space–time domain $\Omega_T \equiv \Omega \times [0, T]$. Conditional stability of this scheme is proved. The decomposition (partition) of the domain into finite elements in the one- and two-dimensional space is shown as well as error estimates of the approximate solutions of the mixed problems are given. In [6], mixed scheme (combinations of FEM, least square method in the space variables and implicit difference scheme in time) is constructed and the stability of the resulting scheme and error estimates of the approximate solutions of the mixed problems are proved. The work [7] is devoted to the study of hyperbolic systems and special attention is paid to the Euler equations. By the use of finite element scheme on the space variables and finite-difference scheme on time, the explicit and implicit schemes are constructed in one-dimensional and two-dimensional regions.

In [8], discontinuous Galerkin method is used to solve one-dimensional transient hyperbolic problems and the local error on each element is shown to be proportional to the Radau polynomial. The discontinuous Galerkin error estimates under

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investigation are computed by solving a local steady problem on each element. For smooth solutions, these error estimates at a fixed time t converge to the true spatial error in the L_2 norm under mesh refinement.

The work [9] presents a finite element approximation of the scalar hyperbolic wave equation written in the mixed form that is, introducing an auxiliary vector field to transform the problem into a first-order problem in space and time. They explained why the standard Galerkin method is inappropriate to solve this problem, and proposed as alternative a stabilized FEM that can be cast in the variational multi scale framework. The unknown is divided into two finite element components and a remainder, referred to as subscale. As original features of their approach, they consider the possibility of letting the subscales to be time dependent and orthogonal to the finite element space.

This paper is devoted to:

- Cover two-dimensional simply connected domains by uniform grid.
- Construct an implicit finite element scheme for the mixed problem for the symmetric t-hyperbolic systems with dissipative boundary conditions.
- Investigate the stability of the resulting scheme.
- The application of the resulting scheme to numerical solution of the mixed problem for symmetric t-hyperbolic system in the simply connected region.

1. Mixed problem (Godunov [1]–Blokhin and Aloev [10])

Let us consider the symmetric t-hyperbolic system

$$A\frac{\partial u}{\partial t}+B\frac{\partial u}{\partial x}+C\frac{\partial u}{\partial y}+Du=F(x,y,t), \tag{1}$$

on the area $G=\{(t,x,y)\,|t\in(0,T)\,,(x,y)\in\Omega\}$ where $\Omega=\{(x,y)\,|\,0<x<l_x,\ |y|<\infty\}$ with the following boundary conditions on $\partial\Omega$ (here $\partial\Omega$ is the boundary of the Ω):

At $x=0$:

$$u^{\mathrm{I}}=Su^{\mathrm{II}},\quad 0<t\le T,\qquad |y|<\infty. \tag{2}$$

At $x=l_x$:

$$u^{\mathrm{I}}=Ru^{\mathrm{II}},\quad 0<t\le T,\qquad |y|<\infty.$$

At $|y|\to\infty$:

$$\|u\|\to 0,\quad 0<t\le T,\ 0\le x\le l_x,\qquad \|u\|=\sqrt{\sum_{k=1}^{N}u_k^2}$$

and initial conditions

$$u\,(0,x,y)=u_0\,(x,y)\,,\quad (x,y)\in\Omega, \tag{3}$$

where x,y,t are independent variables and $A=(a_{ks})\,,\ B=(b_{ks})\,,\ C=(c_{ks})$ are the symmetric constant matrices with dimension $N\times N$, additionally, A is positively defined matrix, $D=(d_{ks})$ is an any $N\times N$ matrix and R,S are rectangular matrices with number of rows equal to appropriate numbers of positive or negative eigenvalues of matrix $B=(b_{ks})$.

In Eqs. (1)–(3), the function $u(x,y,t)=(u_1,u_2,\dots,u_N)^T$ is an unknown vector function to be determined and $F(x,y,t)=(f_1,f_2,\dots,f_N)^T$ is the given vector function dependent on x,y,t.

1.1. Approximation of the area Ω

Let us take a rectangular coordinate system Oxy and draw the straight lines

$$x=x_i=h_xi,\quad \left(i=0,\dots,N_x,h_x=\frac{l_x}{N_x}\right),\qquad y_j=h_yj,\quad j=0,\pm1,\dots,\pm\infty.$$

As a result, the area Ω is covered by a uniform grid. Point of intersection of the straight line $x=x_i$ with $y=y_j$ is called the nodes of the grid and denoted by $M_{ij}=M\left(x_i,y_j\right)$.

Definition 1.1. A node that lies on the boundary $\partial\Omega$ of Ω is called a boundary node.

Definition 1.2. Node lying inside the area Ω and having at least one neighbor boundary node are called nearby boundary nodes.

Definition 1.3. Node lying inside the area Ω and not having neighbor boundary nodes is called inner boundary nodes.