Computers and Mathematics with Applications 68 (2014) 1205–1219

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Convergence analysis of hybrid expanded mixed finite element method for elliptic equations

Huailing Song, Lijian Jiang*, Gaojie Chen

College of Mathematics and Econometrics, Hunan University, Changsha, 410082, China

ARTICLE INFO

Article history:
Received 27 February 2014
Received in revised form 5 August 2014
Accepted 18 August 2014
Available online 15 September 2014

Keywords:
Expanded mixed finite element method
Lagrange multipliers
Lifting operators

ABSTRACT

In this article, we utilize the approach of the variational characterization of the Lagrange multipliers in Cockburn and Gopalakrishnan (2005) and present the convergence analysis of hybrid expanded mixed finite element method for elliptic equations. The unknowns in the expanded mixed formulation can be expressed in terms of lifting operators. The analysis provides a novel perspective for error estimates of the hybrid expanded mixed finite element method. Numerical examples are presented to confirm the convergence analysis.

1. Introduction

To simulate flows in porous media and other applications, it is necessary to use suitable numerical discretization methods retaining local conservation of mass. Mixed finite element method (FEM) is such an approach and has been widely used in science and engineering. However, in many practical situations such as groundwater hydrology and reservoir models in porous media, whose formulations are often created by complex geological processes and may contain materials with a very low ability to transmit fluids, there exist almost impermeable barriers in some regions of the physical domain. For this practical case, its reciprocal is not readily available for use in the standard mixed FEM formulation, and thus, the direct application of the standard mixed FEM to those practical problems may cause some limitations. The expanded mixed FEM extends the standard mixed formulation in the sense that three variables are explicitly approximated, namely, the scalar unknown (e.g., pressure), its gradient and the velocity. The expanded mixed FEM is suitable for the cases where the diffusion coefficient or permeability of the underlying partial differential equations is very small and even partially vanishing inside cells [1], and thus it extends the applications of standard mixed FEM.

Expanded mixed FEM has been extensively studied using traditional finite element analysis. The expanded mixed method was proposed by Wheeler et al. in [2]. Chen [3] analyzed expanded mixed FEM for second-order elliptic equations, and obtained optimal error estimates for linear elliptic equations and certain nonlinear equations. Woodward et al. [4] carried out an error analysis of expanded mixed FEM using the lowest-order Raviart–Thomas space for Richards equation. In [5] Arbogast et al. established the connection between the expanded MFEM and a cell-centered finite difference method by incorporating certain quadrature rules. Efficient solver techniques for the expanded FEM have been discussed in [6,7]. A posteriori error estimator was constructed in [8] and showed more robust behavior than the estimator of standard mixed FEM for the problems with large aspect ratios in diffusion coefficients. Jiang et al. developed expanded mixed multiscale finite element methods using multiscale basis function in the recent work [1], which can work well for flow problems with some permeability fields in shale barriers.

* Corresponding author.
E-mail addresses: shling@hnu.edu.cn (H. Song), ljjiang16@hotmail.com, ljjiang@hnu.edu.cn, ljjiang16@gmail.com (L. Jiang), jchen@hnu.edu.cn (G. Chen).

http://dx.doi.org/10.1016/j.camwa.2014.08.017
0898-1221/

In this paper, we extend the idea in [9,10] to expanded mixed FEM. The idea is to characterize the approximation solution using lifting operators. We use the approach and provide a convergence analysis for the hybrid expanded mixed FEM. This allows the error estimate of the Lagrange multipliers (interface pressure) without using the estimates for the other variables, as is done in classical convergence analysis [5,3,11]. The analysis may shed a light upon the relationship between the four unknown variables in the hybrid expanded mixed formulation: cell pressure, interface pressure, gradient of pressure and the velocity. The new convergence analysis shows that less stringent regularity is required for diffusion coefficient than classical analysis.

The rest of the paper is organized as follows. In Section 2, we present the preliminaries and notations, which will be used in the whole paper. At the same time, the hybrid expanded mixed formulation is presented. In Section 3, we introduce the lifting operators, which are used to characterize the unknown variables. Continuity of the lifting operators is discussed. We provide a variational characterization of the Lagrange multipliers. This is critical for the analysis of the method. Section 4 is devoted to the error estimates for hybrid expanded mixed FEM. In Section 5, numerical experiments are carried out to confirm the convergence analysis and demonstrate the performance of the method. Finally, some comments and conclusions are made.

2. Hybrid expanded mixed formulation

In the paper, we consider the following diffusion equation:

$$\begin{cases} -\mathbf{div}\Big(a(x)\mathbf{grad}p\Big) = f, & \text{on } \Omega, \\ p = g, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbf{R}^N$ is a polyhedral domain $(N \geq 2), f \in L^2(\Omega), g \in H^{\frac{1}{2}}(\partial\Omega)$. For analysis, we assume that the diffusion coefficient $a(x)$ is a symmetric, positive definite matrix.

We introduce the notations used in the paper. For a subdomain D of $\Omega, m \geq 0$ and $1 \leq p \leq \infty, W^{m,p}(D)$ and $L^p(D)$ denote the usual Sobolev space and Lebesgue space, respectively. The norm and seminorm of $W^{m,p}$ are denoted by $\|\cdot\|_{m,p,D}$ and $|\cdot|_{m,p,D}$, respectively. When $p = 2$, $W^{m,p}(D)$ is written as $H^m(D)$ with norm $\|\cdot\|_{m,D}$ and seminorm $|\cdot|_{m,D}$. The norm of $L^2(D)$ is denoted by $\|\cdot\|_{0,D}$. The space $H(div, D)$ is defined as follows,

$$H(div, D) := \{v \in [L^2(D)]^N, \qquad \|v\|_{div,D} := \|v\|_{0,D} + \|\mathbf{div}v\|_{0,D} < \infty\}.$$

We introduce $\theta = \mathbf{grad}p$, $u = -a(x)\theta$. Then Eq. (2.1) can be rewritten as

$$\begin{cases} u + a(x)\theta = 0 \\ \theta - \mathbf{grad}p = 0 \\ \mathbf{div}u = f. \end{cases} \tag{2.2}$$

Here u physically refers to the fluid velocity. We define the following function spaces for solutions:

$$X_1 = [L^2(\Omega)]^N, \qquad X_2 = H(div, \Omega), \qquad Q = L^2(\Omega).$$

The inner products associated with these spaces can be straightforwardly defined. We use the notation $(\cdot,\cdot)_{X_1}$ to denote the inner product in X_1, and use similar notations for the inner products in other spaces. The notation $(\cdot,\cdot)$ denotes the usual L^2 inner product. The expanded mixed formulation of Eq. (2.1) reads: find (θ, u, p) in $X_1 \times X_2 \times Q$, satisfying

$$\begin{cases} (u, v) + (a(x)\theta, v) = 0, \quad \forall v \in X_1 \\ (\theta, w) + (p, \mathbf{div}w) = (g, w\cdot n)_{\partial\Omega}, \quad \forall w \in X_2 \\ (\mathbf{div}u, q) = (f, q), \quad \forall q \in Q. \end{cases} \tag{2.3}$$

Here the n denotes the outward unit normal to Ω.

Let $\mathfrak{T}_h$ be a quasi-uniform partition of Ω and K be a representative element with $diam(K) = h_K$. Let $h = \max\{h_K, K \in \mathfrak{T}_h\}$. Let (V_h, Q_h) be a classic mixed finite element space pair such as the Raviart–Thomas, Brezzi–Douglas–Marini (BDM) or Brezzi–Douglas–Fortin–Marini spaces (cf. [12]). We define $X_1^h \times X_2^h \times Q^h$ to be a finite element space associated with $X_1 \times X_2 \times Q$,

$$X_1^h = \{v \in [L^2(\Omega)]^N : \ v|_K \in V_h(K) \text{ for all } K \in \mathfrak{T}_h\},$$
$$X_2^h = \{w \in H(div, \Omega) : \ w|_K \in V_h(K) \text{ for all } K \in \mathfrak{T}_h\},$$
$$Q^h = \{q \in L^2(\Omega) : \ q|_K \in Q_h(K) \text{ for all } K \in \mathfrak{T}_h\}.$$

For the convergence analysis, we assume that (V_h, Q_h) is the Raviart–Thomas finite element space pair in the paper. The expanded mixed FEM formulation of Eq. (2.3) reads: find $(\theta_h, u_h, p_h) \in X_1^h \times X_2^h \times Q^h$ satisfying

$$\begin{cases} (u_h, v) + (a(x)\theta_h, v) = 0, \quad \forall v \in X_1^h \\ (\theta_h, w) + (p_h, \mathbf{div}w) = (g, w\cdot n)_{\partial\Omega}, \quad \forall w \in X_2^h \\ (\mathbf{div}u_h, q) = (f, q), \quad \forall q \in Q^h. \end{cases} \tag{2.4}$$