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Positive solutions of a Lotka–Volterra competition model with cross-diffusion☆

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ABSTRACT

This paper concerns a Lotka–Volterra competition reaction–diffusion system with nonlinear diffusion effects. We first briefly discuss the stability of trivial solutions by spectrum analysis. Based on the boundedness of the solutions, the existence of the positive steady state solution is also investigated by the monotone iteration method.

1. Introduction

In recent decades, there has been rapidly growing interests in the analysis and modeling of biological system. The overall behavior of ecological systems continue to be of great interest to both applied mathematicians and ecologists. From the view of human needs, the exploitation and utilization of biological resources of different species are frequently practiced in many fields. There is a considerable literature on dynamical systems modeling some species by applying differential equations. The dynamic behavior of ecological systems will continue to be of interest to both applied mathematicians and ecologists due to its universal existence and importance.

The diffusion phenomenon of different species in the environment is a very universal survival and life style. During the past years, the reaction–diffusion systems derived from interactions of several species have been extensively studied. Among those systems, the Lotka–Volterra model is a very important researching branch. The Lotka–Volterra equations have been known as the basis of many types of models that involve the interactions of different populations, some concepts such as diffusion and functional response have been added to the Lotka–Volterra equations to gain a better understanding of the dynamics of population interactions. Many complex models for two or more interacting species have been proposed on the basis of Lotka–Volterra models by taking into account the effects of diffusion, age structure, time delay, functional response, etc.

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In mathematical ecology, the classical dynamical system, due independently to Lotka and Volterra in the 1920s, reflects only population changes due to predation in a situation where the species densities are not spatially dependent. It does not take into account either the fact that population is usually not homogeneously distributed, nor the fact that the species naturally develop strategies for survival. Both of these considerations involve diffusion processes which can be quite intricate as different concentration levels of species cause different population movements. Such movements can be determined by the concentration of the same species (diffusion) and that of other species (cross-diffusion).

The diffusion behavior in different species not only affect the survival of the individual and the species but the distribution of resources also. The effect of diffusion on the resource-biomass is sometimes great. It may lead to the environment changing, and in extreme cases the extinction of species due to diffusion may also occur affecting the bio-diversity of the ecosystem. Therefore, the consideration of diffusion effect is very logical and close to real systems.

In recent years, there has been considerable interest in being able to reveal the dynamics of strongly coupled reaction–diffusion systems with cross-diffusion. It should be pointed out that most discussions have concentrated on the Lotka–Volterra interaction system which was proposed first by Shigesada et al. in [1]. For instance, refer to [2–8] and the references therein.

Keeping these in view, in the present paper, a differential model with cross-diffusion is proposed to study the effects of diffusion on a biological species as well as on its resource-biomass. We investigate the following Lotka–Volterra competition reaction–diffusion system with nonlinear diffusion effects

$$\begin{cases} u_t - \Delta\left(u\left(d_1 + \alpha_1 v + \dfrac{\gamma_1}{\beta_1 + v}\right)\right) = u(a_1 - b_1 u - c_1 v), & x \in \Omega,\ t > 0, \\ v_t - \Delta\left(v\left(d_2 + \alpha_2 u + \dfrac{\gamma_2}{\beta_2 + u}\right)\right) = v(a_2 - b_2 u - c_2 v), & x \in \Omega,\ t > 0, \\ u = v = 0, \quad x \in \partial\Omega,\ t > 0, \\ u = u_0 \geq, \not\equiv 0, \qquad v = v_0 \geq, \not\equiv 0, \quad x \in \Omega,\ t = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $u = u(x, t)$, $v = v(x, t)$ are the population densities of two species. $a_i, b_i, c_i, d_i, i = 1, 2$, are positive constants, $\alpha_i, \beta_i, \gamma_i, i = 1, 2$, are nonnegative constants, a_i and d_i represent the birth rates and diffusion coefficients of u, v respectively, α_i and β_i are the cross-diffusion coefficients and cross-diffusion pressures. The nonlinear term $\alpha_i\Delta(uv)$ is usually referred as the cross-diffusion term, the nonlinear term of fractional form $\Delta(\frac{\gamma_1 u}{\beta_1+v})$ (resp. $\Delta(\frac{\gamma_2 v}{\beta_2+u})$) models the situation in which the chaseable capacity of the species u (resp. v) is decreasing with the enhanced resistance of the species v (resp. u). *The system (1.1) means that, in addition to the dispersive force, the diffusion also depends on population pressure from other species.*

The reaction–diffusion systems with linear diffusion effects have been investigated widely, and many valuable results have been obtained (see [9–18] for example). However, similar to such fractional-nonlinear diffusion terms in (1.1) in the field of reaction–diffusion systems are not very many. In [19], Kadota and Kuto discussed a prey–predator system with strongly coupled and fractional-nonlinear diffusion terms. Using bifurcation theory and a priori estimates for steady state solutions, they gave a sufficient condition for the existence of positive steady state solutions for the system. For more specific and detailed backgrounds of cross-diffusion systems, one can refer to [1–8,20–25] and the references therein.

The main aim of this paper is to study the steady state system of (1.1), that is, the classical positive solutions of the following elliptic system

$$\begin{cases} -\Delta\left(u\left(d_1 + \alpha_1 v + \dfrac{\gamma_1}{\beta_1 + v}\right)\right) = u(a_1 - b_1 u - c_1 v), & x \in \Omega, \\ -\Delta\left(v\left(d_2 + \alpha_2 u + \dfrac{\gamma_2}{\beta_2 + u}\right)\right) = v(a_2 - b_2 u - c_2 v), & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.2}$$

For system (1.2), we mainly discuss the following: the boundedness and the existence of positive solution. For the existence of positive solution, our approach to the proof is based on the monotone iteration method and some ideals of Pao in [17,26]. We prove that system (1.2) admits at least or only one positive solution on certain conditions.

If $\alpha_i, \gamma_i \equiv 0, i = 1, 2$, then (1.2) becomes the classical Lotka–Volterra competition system.

This paper is organized as follows. In Section 2, we briefly discuss the stability of trivial and semi-trivial solutions of (1.2). Section 3 concerns the boundedness of positive solution of (1.2). In Section 4, we first present the sufficient conditions which ensure (1.2) has no positive solution. And then, we mainly investigate the existence of positive steady state solutions of (1.1). The methods of nonlinear analysis and the tools of nonlinear partial differential equations that we used in the present paper are somewhat useful for different readers in applied subjects.

For convenience, we first give some preliminaries (see [15] for example).

For $q(x) \in C(\overline{\Omega})$, we denote $\lambda_1(q)$ by the principal eigenvalue of the problem

$$-\Delta u + qu = \lambda u, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega. \tag{1.3}$$