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A regularized smoothing Newton-type algorithm for quasi-variational inequalities ☆

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ABSTRACT

This paper is concerned with developing an efficient regularized smoothing Newton-type algorithm for quasi-variational inequalities. The proposed algorithm takes the advantage of newly introduced smoothing functions and a non-monotone line search strategy. It is proved to be globally and locally superlinearly/quadratically convergent under suitable assumptions. Numerical results demonstrate that the algorithm generally outperforms the existing interior point method and semismooth method (Facchinei, et al. 2014).

1. Introduction

In this paper, we consider the finite-dimensional quasi-variational inequality, denoted by QVI, i.e., find a point $x^* \in K(x^*)$ such that

$$F(x^*)^T(y - x^*) \geq 0, \quad \forall y \in K(x^*), \tag{1.1}$$

where $F : \Re^n \to \Re^n$ is a (point-to-point) mapping and $K : \Re^n \rightrightarrows \Re^n$ is a (point-to-set) mapping with closed, convex images and

$$K(x) := \{y \in \Re^n | g(y, x) \leq 0\},$$

where $g : \Re^n \times \Re^n \to \Re^m$. We assume that $g_i(\cdot, x)$ is convex and continuously differentiable on $\Re^n$, for each $x \in \Re^n$ and each $i = 1, \ldots, m$. In particular, when the set K is independent of x, QVI (1.1) reduces to the well-known variational inequality, denoted by VI, i.e., determine a point x^* in a closed convex subset K of $\Re^n$ such that

$$F(x^*)^T(y - x^*) \geq 0, \quad \forall y \in K.$$

The reader is referred to [1,2] and the references therein.

QVI (1.1) was firstly introduced by Bensoussan and Lions [3]. This model has important applications in generalized Nash games, mechanics, economics, statistics, transportation, and biology (see, for example, [1,4–6]). Among them, one of the

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main task is to investigate how to solve QVI (1.1). In this paper, we mainly focus on the numerical reconstruction method based on the KKT conditions of QVI (1.1). Recently, an interior point method was proposed for solving QVI (1.1) [6], which includes many classes of QVI (1.1). It established a global convergence result based on a potential reduction method. Very recently, a so called LP-Newton method was proposed for solving the nonsmooth systems of equations with nonisolated solutions [7,8]. [4] further investigated the methods given in [7,8] and analyzed the locally fast convergence for generalized Nash equilibrium problems as special cases of QVI (1.1). Lately, a semismooth Newton method was proposed for solving quasi-variational inequalities [5]. By exploiting Burmeister–Fischer *NCP-function* (see (2.3)) which is useful for reformulating the KKT conditions of QVI (1.1), it obtained global convergence and locally superlinear/quadratic convergence result for some important classes of quasi-variational inequality problems. Very encouraging numerical results were also reported in [5].

Nearly thirty years of time, smoothing Newton-type algorithm has been extensively applied to solving various optimization problems (see, for example, [9–17]). The main idea of this class of algorithms is to reformulate the problem concerned as a family of parameterized smooth equations and then to solve the smooth equations approximately by using Newton-type methods at each iteration. By driving the parameter to zero, one expects that a solution to the original problem can be found. It should be noted that this class of methods, in essence, is an iterative method. Two key points of iterative method are to find a direction and a stepsize. So finding an appropriate direction and a stepsize plays a crucial role in designing a good iterative method.

It is well known that smoothing Newton-type algorithm has the following distinguishing features. Firstly, the algorithm can start from an arbitrary initial point. Secondly, the algorithm needs to solve only one linear system of equations and performs only one suitable line search at each iteration. Thirdly, it possesses not only good numerical performance, but also the global and local superlinear/quadratic convergence under some assumptions.

Based on significant advantages stated above, a regularized smoothing Newton-type algorithm is proposed in this paper and applied to solving QVI (1.1) with a new family of smoothing functions. By regularization, we do improve the conditions that nonsingularity of Jacobian matrix $JH(z^k)$ (see (2.7)) holds compared with Theorem 3 given in [6] (see Theorem 3.1 for details). Furthermore, we show that the proposed algorithm is globally and locally superlinearly/quadratically convergent under some suitable assumptions. We also report some preliminary numerical results.

The rest of this paper is organized as follows. In the next section, QVI (1.1) is reformulated as a system of parameterized smooth equations with a new family of smoothing functions, and some essential properties are given with respect to the new family of smoothing functions. In Section 3, we propose a regularized smoothing Newton-type algorithm for solving the KKT system of QVI (1.1) and analyze the solvability of corresponding Newton equations. In Section 4, we discuss the convergence of the proposed algorithm. In Section 5, we report the preliminary numerical results. Some conclusions are drawn in Section 6.

In the following, we introduce the notations that are used. The superscript T denotes the transpose. $\Re^n_+$ (respectively, $\Re^n_{++}$) denotes the nonnegative (respectively, positive) orthant in $\Re^n$. We denote $\mathcal{I} = \{1, 2, \ldots, m\}$. For any vectors $u, v \in \Re^n$, we write $(u^T, v^T)^T$ as (u, v) for simplicity. For any function $F : \Re^m \to \Re^m$, if F is differentiable at x, then $JF(x)$ denotes the Jacobian matrix of F at x and $\nabla F(x)$ denotes the transposed Jacobian. Given a smooth mapping $g : \Re^n \times \Re^n \to \Re^m$, $(y, x) \mapsto g(y, x)$, $\nabla_y g(y, x)$ denotes the transpose of the partial Jacobian of g with respect to the y-variables. If F is locally Lipschitz continuous around x, then $\partial F(x)$ denotes Clarke's generalized Jacobian of F at x. For any $a, b \in \Re_+$, $a = O(b)$ (or $a = o(b)$) means $\limsup_{b\to 0} \frac{a}{b} < +\infty$ (or $\limsup_{b\to 0} \frac{a}{b} = 0$). While k denotes the iterative index, the set of all iterative indices is written as $\mathcal{J}$, i.e., $\mathcal{J} := \{0, 1, 2, \ldots\}$.

2. Smoothing reformulation for quasi-variational inequalities

From the differentiability of function $g(\cdot, x)$, the Karush–Kuhn–Tucker (KKT) conditions of QVI (1.1) are in the following form:

$$
\begin{aligned}
&F(x) + \nabla_y g(x, x)\lambda = 0,\\
&\lambda \geq 0, \quad g(x, x) \leq 0, \quad \text{and} \quad \lambda^T g(x, x) = 0,
\end{aligned} \tag{2.1}
$$

where $\lambda \in \Re^m$ are Lagrange multipliers. From Theorem 1 of [6], it is not difficult to see that $x^* \in K(x^*)$ is a solution of QVI (1.1) if and only if there exists $\lambda^* \in R^m$ such that (x^*, λ^*) satisfies the KKT conditions (2.1).

Let $L(x, \lambda) := F(x) + \nabla_y g(x, x)\lambda$, $h(x) := g(x, x)$, then (2.1) can be equivalently rewritten as

$$
\begin{aligned}
&L(x, \lambda) = 0,\\
&h(x) + w = 0,\\
&\lambda \geq 0, \quad w \geq 0, \quad \text{and} \quad \lambda^T w = 0,
\end{aligned} \tag{2.2}
$$

where $w \in \Re^m$ are slack variables. It is easy to see that (2.2) contains a complementarity system. There may be a certain degree of difficulty in solving (2.2) directly. From [2] we can see that the complementarity system can be replaced by *NCP-function*. For *NCP-function*, the normal definition is given as follows.

Definition 2.1. Function $\psi : \Re^2 \to \Re$ is called a NCP-function, if for any $(a, b)^T \in \Re^2$, $\psi(a, b) = 0 \;\Leftrightarrow\; a \geq 0, b \geq 0, ab = 0$.