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High order parameter-uniform discretization for singularly perturbed parabolic partial differential equations with time delay

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ABSTRACT

In this article we study numerical approximation for singularly perturbed parabolic partial differential equations with time delay. A priori bounds on the exact solution and its derivatives, which are useful for the error analysis of the numerical method are given. The problem is discretized by a hybrid scheme on a generalized Shishkin mesh in spatial direction and the implicit Euler scheme on a uniform mesh in time direction. We then design a Richardson extrapolation scheme to increase the order of convergence in time direction. The resulting scheme is proved to be second order accurate in time direction and fourth order (with a factor of logarithmic type) accurate in spatial direction. Numerical experiments are performed to support the theoretical results.

1. Introduction

Delay differential equations (DDEs) model a large variety of practical phenomena in Biosciences, for example, in population dynamics and epidemiology, where the delay is due to a gestation or maturation period, or, in engineering and numerical control, where the delay arises from the processing in the controller feedback loop, in which the time evolution depends not only on present states but also on states at or near a given time in the past (see, e.g., [1,2]). If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small parameter, then it is said to be the class singularly perturbed delay differential equations (SPDDEs). Such problems arise in mathematical modeling of various practical phenomena, for example, in population dynamics [3], the study of bistable devices [4], description of the human pupil–light reflex [5], and variational problems in control theory [6].

Singularly perturbed delay differential equations have been studied extensively (and almost exclusively) in the context of ordinary differential equations (ODEs). While singularly perturbed delay PDEs are less well understood. They are typically of the form

$$\partial_t u(x,t) = Lu(x,t,u_{(x,t)}) + f(x,t), \tag{1}$$

where $u_{(x,t)}$ is a function segment, which can extend both in the past and over some region in space: $u_{(x,t)}(r,s) = u(r+x, s+t)$, $(r,s) \in [-\tau, 0] \times [-\sigma, \sigma]$. Eq. (1) has to be completed with boundary conditions and an initial condition, which

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typically have to be specified over some initial and boundary regions around the domain of definition of the delay PDEs. A characteristic example from numerical control is the following equation

$$(\partial_t u - \varepsilon \partial_x^2 u)(x, t) = v(g(u(x, t - \tau)))\partial_x u(x, t) + c[f(u(x, t - \tau)) - u(x, t)], \quad (x, t) \in D \tag{2}$$

which models a furnace used to process metal sheets. Here, u is the temperature distribution in a metal sheet, moving at a velocity v and heated by a source specified by the function f; both v and f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length τ. A set of examples, illustrating the wide range of existing delay PDE models can be found in Wu [2].

In the direction of numerical study of singularly perturbed partial differential equations with time delay, much can be seen in [7–10], and the references therein. In particular for our model problem (3)–(5), the authors in [7,8] designed parameter uniform numerical methods using the fitted mesh and fitted operator approach, respectively, which results in uniform convergence of first order in time and second order in spatial direction. High order numerical methods are of great interest for the numerical community. They are fairly understood for singularly perturbed problems without delay, see [11–14] and the references therein. Nevertheless, so far we do not know any paper, having order of uniform convergence more than one in time direction and two in spatial direction, for singularly perturbed delay parabolic problem (3)–(5). Therefore, in the present paper, our objective is to design and analyze a high order parameter-uniform numerical method for this problem. As a first attempt at designing a high order numerical method, we employ a hybrid scheme of HODIE type (see [15]) on a generalized Shishkin mesh in spatial direction and implicit Euler scheme on uniform mesh in time direction. Then we design a Richardson extrapolation scheme to increase the order of uniform convergence in time direction. We prove that the final computed solution is uniformly convergent of $O((L/N)^4 + (\Delta t)^2)$. We also consider nonlinear singularly perturbed delay parabolic PDEs. The quasilinearization technique is employed to reduce the original nonlinear problem into a sequence of linear problems, each of which is then solved by the method developed for the linear case.

A description of the contents of the article is as follows. The model problem is formulated in Section 2. Assumptions for the existence, uniqueness and appropriate regularity of the solutions to the problem are then presented. Also, some a priori bounds on the exact solution and its derivatives of the model problem (3)–(5) are given. In Section 3 the problem is discretized by a hybrid scheme on a generalized Shishkin mesh in spatial direction and implicit Euler on a uniform mesh in time direction. In Section 4 we proved that the method is uniformly convergent of $O((L/N)^4 + \Delta t)$. The Richardson extrapolation scheme is designed to increase the convergence of the numerical method in time direction in Section 5. We then proved that the order of uniform convergence of the resulting numerical method increases to $O((L/N)^4 + (\Delta t)^2)$. In Section 6, we consider a nonlinear singularly perturbed delay parabolic PDE and employ the quasilinearization technique to resolve the nonlinearity of the problem. The resulting sequence of linear singularly perturbed delay parabolic PDE can be solved by the method developed for the linear case. Finally, numerical results are presented in Section 7 to validate our theoretical findings.

Notation. Throughout the paper C is a generic positive constant that is independent of ε and discretization parameters. For any function $g \in C([0, 1] \times [0, T])$, define $g_{i,j} = g(x_i, t_j)$. We consider the maximum norm and denote it by $\| \cdot \|_S$, where S is a closed and bounded subset of $[0, 1] \times [0, T]$. When the domain is obvious, or of no particular significance, S is usually omitted. The analogous discrete maximum norm on the mesh S^{N,N_t} is denoted by $\| \cdot \|_{S^{N,N_t}}$.

2. Singularly perturbed delay parabolic PDEs

We consider the following singularly perturbed parabolic problem with time delay

$$\mathcal{L}u(x, t) := \partial_t u(x, t) - \varepsilon \partial_x^2 u(x, t) + b(x, t)u(x, t) = -a(x, t)u(x, t - \tau) + f(x, t) \quad (x, t) \in D \tag{3}$$

$$u(x, t) = \gamma_\ell(t), \quad (x, t) \in \Gamma_\ell, \qquad u(x, t) = \gamma_r(t), \quad (x, t) \in \Gamma_r, \tag{4}$$

$$u(x, t) = \gamma_b(x, t), \quad (x, t) \in \Gamma_b, \tag{5}$$

where $0 < \varepsilon \ll 1$, $D = \Omega \times (0, T] = (0, 1) \times (0, T]$, Γ_ℓ and Γ_r are left and right hand sides of the domain D corresponding to $x = 0$ and $x = 1$, respectively, and $\Gamma_b = [0, 1] \times [-\tau, 0]$, $\tau > 0$. For simplicity we consider $T = n\tau$ for some integer $n > 1$. The notations $\widetilde{\Gamma} = \Gamma_\ell \cup \Gamma_r \cup \Gamma_b$ and $\Gamma = \Gamma_\ell \cup \Gamma_r \cup \Gamma_0$, where $\Gamma_0 = [0, 1] \times \{0\}$, are also used. The reaction term b and the delay term a are assumed to be sufficiently smooth and bounded that satisfy

$$b(x, t) \geq \beta > 0, \qquad a(x, t) \geq \alpha > 0, \quad (x, t) \in \overline{D}. \tag{6}$$

For small ε, it is clear that the solution of (3)–(5) has boundary layers on Γ_ℓ and Γ_r. The characteristics of the reduced problem of (3)–(5) (after putting $\varepsilon = 0$) are the vertical lines $x = constant$, which implies that any boundary layers arising in the solution are of parabolic type.

The existence of a unique solution of (3)–(5) is shown in [7,16], with the assumption that the data are smooth and satisfy appropriate compatibility conditions at the corner points $(0, 0)$, $(1, 0)$, $(0, -\tau)$ and $(1, -\tau)$. The required compatibility conditions are

$$\gamma_b(0, 0) = \gamma_\ell(0), \qquad \gamma_b(1, 0) = \gamma_r(0), \tag{7}$$