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On a time and space discretized approximation of the Boltzmann equation in the whole space

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ABSTRACT

In this paper, convergence results on the solutions of a time and space discrete model approximation of the Boltzmann equation for a gas of Maxwellian particles in a bounded domain, obtained by Babovsky and Illner (1989), are extended to approximate the solutions of the Boltzmann equation in the whole physical space. This is done for a class of particle interactions including Maxwell and soft cut-off potentials in the sense of Grad.

The main result shows that the solutions of the discrete model converge in $\mathbb{L}^1$ to the solutions of the Boltzmann equation, when the discretization parameters go simultaneously to zero. The convergence is uniform with respect to the discretization parameters.

In addition, a sufficient condition for the implementation of the main result is provided. The techniques detailed in this paper may be also applied in other contexts.

1. Introduction

In a known paper [1], Babovsky and Illner provided a validation (convergence) proof of Nanbu's simulation method [2] for the spatially inhomogeneous (full) Boltzmann equation [3,4] describing a rarefied gas of Maxwellian particles confined to a bounded spatial domain (with specularly reflecting boundary conditions). More specifically, the main result (Theorem 7.1) of [1] demonstrated that the discrete measures provided by Nanbu's simulation method are almost surely weakly convergent to absolutely continuous measures with densities given by solutions of the Boltzmann equation.

In essence, the analysis behind the main theorem of [1] represented a space-dependent generalization of the convergence proof of Nanbu's simulation algorithm for the space-homogeneous Boltzmann equation, provided in an earlier work by Babovsky [5]. Briefly, in [1], the space-homogeneous simulation algorithm of [5] was applied to a suitable time and space discrete Boltzmann model (Eq. (5.14) in [1]). The latter was derived from the Boltzmann equation by means of time discretization, splitting (separation of free flow and collisional interactions), cell-partitioning of the physical domain of the gas, and space-averaging (homogenization) over cells. The discretization was parameterized by a time-step and an upper bound for the maximum of all cell diameters. The analysis was completed by combining convergence properties of the discrete Boltzmann model with those of the space-homogeneous simulation algorithm of [5]. To this end, Babovsky and Illner established the key result (Corollary 5.1 in [1]) that the solutions of the discrete Boltzmann model converge in discrepancy[1] to the solutions of the Boltzmann equation for the gas in a bounded spatial volume, uniformly with respect to the parameters of the discretization, when these parameters converge simultaneously to zero.

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[1] Let μ and ν be two (Borel) probability measures on the same measure space $\mathfrak{B} \subseteq \mathbb{R}^n$. Consider $\mathfrak{B}$ with the usual semi-order $\leq$ of $\mathbb{R}^n$. Following [1], the discrepancy between μ and ν is defined as $D(\mu, \nu) := sup_{z\in\mathbb{R}^n} |\mu\{y \in \mathfrak{B} : y \leq z\} - \nu\{y \in \mathfrak{B} : y \leq z\}|$.

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A notable thing about the proof of the convergence in discrepancy of the solutions of the discretized Boltzmann model is that, as it appears in [1], is responsible for the limitation of the analysis of [1] to the case of the Boltzmann gas in a bounded spatial domain. Indeed, the boundedness of the spatial domain was actually assumed in [1] in order to prove the above key convergence result for the discrete Boltzmann equation (see [1, p. 59]). An alternative proof, without the boundedness assumption, might allow the conclusions of [1] to be extended to other important examples, e.g., a gas expanding in the whole physical space.

In this paper, the results on the convergence in discrepancy established by Babovsky and Illner for the solutions of their discrete Boltzmann model of [1] are extended to the setting of the Boltzmann equation in the entire physical space. More specifically, in such a setting, we show that the solutions of the discrete Boltzmann model converge in $\mathbb{L}^1$ to the solutions of the Boltzmann equation in the whole physical space, uniformly with respect to the parameters of the discretization, when these parameters converge simultaneously to zero. We also show that the solutions of the discrete approximation satisfy the conservation laws for mass, momentum and energy.

Here, it should be recalled that the results of [1] concern the Boltzmann equation for Maxwellian particles. The limitation to Maxwellian interactions does not come from the proof of the analytical convergence of the discretized Boltzmann model, but is imposed by the implementation of the simulation algorithm of [5] for the validation of Nanbu's scheme (see [1, p. 48]). However, besides its usefulness in the validation of the Nanbu's scheme, the discrete Boltzmann model of [1] might be applied to obtain new (not necessarily probabilistic) rigorous algorithms for the Boltzmann equation. Thus, understanding its convergence properties in more general situations than in [1] may be of interest. In this respect, as an additional contribution, our main result concerns the Boltzmann equation with Maxwell and soft cut-off collision kernels in the sense of Grad [6].

Compared to [1], our analysis must face additional difficulties, since one has to estimate, uniformly, in some sense, how high speed gas particles situated at large distances contribute to the gas evolution, approximated as in [1], by an alternation of molecular transport and collision steps. In this respect, a technical point is reconsidering the important property established by Babovsky and Illner (Theorem 5.1 in [1]) that, under suitable conditions, if the Boltzmann equation is approximated by the discrete Boltzmann model, then the family of errors introduced by the approximation is bounded in some $\mathbb{L}^\infty$-(velocity) Maxwellian weighted space, uniformly with respect to the parameters of the discretization. This property was demonstrated in [1], in the setting of the Boltzmann equation in a bounded spatial domain, but remains actually valid in a larger context, as is implicit from [1]. Nevertheless, for the sake of clarity and completeness, in the present work, we will prove a precise statement appropriate to our framework (see Proposition 1 in Section 4.2).

The techniques of this paper can be also applied to the mixed problem for the Boltzmann equation in a bounded volume, considered in [1]. The main advantage is that convergence results of [1] are reobtained, independently of the assumption of boundedness of the spatial domain. This allows for applications to the initial–boundary volume problem for the Boltzmann equation in an unbounded spatial domain, at least, for the boundary conditions mentioned in [1] (specular/inverse reflection), and for suitable geometries of the boundary.

Our analysis is developed in the framework of the Boltzmann gas characterized by distribution functions decaying at infinity in position and velocity. In our case, these are $\mathbb{L}^\infty$-Maxwellian weighted spaces. In essence, other spaces of functions with slower decay at infinity can be also used, e.g., $\mathbb{L}^\infty$-weighted spaces with polynomial or polynomial $\times$ Maxwellian weights (as is explained in the last section of this paper). Even so, the condition of decay at infinity is a limitation of the method.

The generalization of the method to hard cut-off collision kernels (including hard sphere) and non cut-off collision kernels remains open.

As already emphasized in [1], our results are actually conditioned by the availability of appropriate results on the existence of solutions for the Boltzmann equation.

From a numerical point of view, we use an explicit scheme which has limitations, because of the boundedness restrictions on the timestep Δt. These are known to be even more inconvenient in the case of non Maxwellian molecules. The scheme appears to be less adequate in steady-state applications, where implicit methods may be more efficient [7].

The rest of this paper is structured as follows. In Section 2, we present the discrete Boltzmann model of [1], and formally introduce the main result. However, a precise formulation (Theorem 1) is given in the second part of Section 3. This requires some preparation in the first part of the same section. The second part of Section 3 also includes Theorem 2 which provides sufficient conditions for the application of Theorem 1. Section 4 deals with the proofs of the theorems stated in Section 3. The proofs rely on technical estimates provided in Section 4.1. In particular, standard $\mathbb{L}^\infty$-type inequalities for the collision term are adapted to our framework, supplemented with useful $\mathbb{L}^1$-estimates. The central result of Section 4.1 is Lemma 4, needed later to measure, in some sense, the errors introduced when the discrete Boltzmann model approximates the Boltzmann equation. The results of Section 4.1 are then used in Section 4.2 to prove Proposition 1, ultimately leading to the proof of Theorem 1. Section 4.3 contains the proof of Theorem 2. Finally, Section 5 presents a simple application to the Boltzmann model for a rarefied gas expanding in the whole space, and discusses briefly the generalizations and applications of the main result of this paper to other contexts.

2. Discretized Boltzmann model for the Boltzmann equation

In this section we recall some very basic facts about the Boltzmann equation, and briefly present its time and space discrete approximation of [1], adapted to our setting. Finally, we formally introduce our results.