Computers and Mathematics with Applications 68 (2014) 13–26

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Obtaining artificial boundary conditions for fractional sub-diffusion equation on space two-dimensional unbounded domains

Rezvan Ghaffari, S. Mohammad Hosseini *

Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-175, Tehran, Iran

ARTICLE INFO

Article history:
Received 11 November 2013
Received in revised form 24 March 2014
Accepted 3 May 2014
Available online 21 May 2014

Keywords:
Fractional sub-diffusion equation
Artificial boundary condition
Two-dimensional unbounded domain
Laplace transform
Modified Bessel equation

ABSTRACT

In this paper, we consider fractional sub-diffusion equation on two-dimensional unbounded domains and obtain an exact and some approximating artificial boundary conditions for it by a joint application of the Laplace transform and the Fourier series expansion. In order to test the derived artificial boundary condition, after reducing the main problem to an initial–boundary value problem on bounded domain by imposing the obtained boundary condition, we just use classical Crank–Nicolson method for space variables and L1 approximation for the fractional time derivative. Some numerical examples are given which confirm the effectiveness of the proposed artificial boundary conditions.

1. Introduction

Fractional differential equations and fractional calculus arise in various application problems in physical, biological, geological, and chemical systems; mechanical engineering, environment sciences, signal processing, systems identification, electrical and control theory, etc., see [1–6]. The fractional sub-diffusion equation is a subclass of anomalous diffusive systems, which can be obtained from the standard parabolic PDEs by replacing the first-order time derivative with a fractional derivative of order $\alpha,\ 0 < \alpha < 1$. For some general results regarding the existence and uniqueness of the solution of fractional differential equations we can refer the reader to [1,7].

We consider fractional sub-diffusion equation on two-dimensional unbounded domains and obtain an exact and some approximating artificial boundary conditions for it by a joint application of the Laplace transform and the Fourier series expansion. As far as we searched the related literature there were no reports on similar results as we are investigating in this paper.

The problem of fractional sub-diffusion equation on bounded domains in two space dimensions has been widely studied from different aspects. There are many numerical methods reported for related problems on bounded domain. For example, explicit and implicit finite difference schemes [8,9], implicit RBF meshless approach [10], alternating direction implicit (ADI) scheme [11,12], high-order compact finite difference [13], orthogonal spline collocation (OSC) [14], Kansa-type method [15], etc.

However, considering such problems on unbounded two-dimensional domains could be very interesting and important in applications. The great difficulties to obtain the numerical solutions of such problems on unbounded physical domains lie

* Corresponding author.
E-mail addresses: r.ghaffari85@gmail.com (R. Ghaffari), hossei_m@modares.ac.ir, hosseini.gm@gmail.com (S.M. Hosseini).

http://dx.doi.org/10.1016/j.camwa.2014.05.006

in the unboundedness of physical domains. There are a lot of papers that have discussed the analytical solutions of fractional problems. For some sort of fractional equations on bounded domains, for example see [16–18] and on unbounded domains using various methods such as Laplace and Fourier transforms, for example see [19–21]. Solutions of such problems are often represented in terms of Fox-H functions, M-Wright functions, Green functions and Mittag-Leffler functions, where these functions are in the form of series and their computation is not so easy. In this work we focus on a novel derivation of artificial boundary conditions, because the fractional problems on unbounded domains in application usually need to be solved numerically with efficient use of any suitable numerical method. However, the concept of uniqueness and continuous dependence of the solution of the fractional sub-diffusion equation with a polar coordinate in two-dimensional unbounded domains, will be investigated in detail in future works. Artificial boundary method (ABM) has been one of the most important and efficient methods for numerical solution of partial differential equations (PDEs) in unbounded domains and has been widely applied for integer order equations on unbounded domains in problems such as wave equations [22–24], parabolic equations [25–27], more different types of equations [28] and one-dimensional fractional sub-diffusion equations [29–31]. In the artificial boundary method, at first, one divides the original unbounded domain by an artificial boundary into subdomains: a bounded domain and an unbounded residual domain. Then the problem in unbounded residual domain is considered for deriving the exact and approximating artificial boundary conditions on the artificial boundary. Second, the obtained artificial boundary condition is used by any suitable numerical methods, newly introduced or from those already reported in the literature over bounded domains, to solve the problem in bounded domain.

So, in this paper, we investigate the derivation of artificial boundary condition for the fractional sub-diffusion equation on space two-dimensional unbounded domains. The results should be very interesting for applications in sciences and engineering. Using the obtained artificial boundary conditions, any of the already introduced numerical methods by various authors for bounded two-dimensional domains can easily be applied to the problem on bounded subdomain. However, to test our findings we just apply the Crank–Nicolson finite difference method without losing net worth of the obtained results.

The discussion in the rest of paper will be as follows. In Section 2, we express the problem in unbounded domain and introduce an artificial boundary. In Section 3, we derive artificial boundary conditions by using the Laplace transformation and some interesting properties of the solutions of the modified Bessel equations. Finally, in Section 4, some numerical results are presented to show the effectiveness of the results proposed in this paper.

2. The problem

Let $D \subset \mathbb{R}^2$ denote a bounded domain where $D = \left\{\mathbf{x} \in \mathbb{R}^2 | \, \|\mathbf{x}\| < a\right\}$. Suppose

$$\Gamma_0^T = \partial D \times [0, T], \qquad D^c = \mathbb{R}^2 \backslash \bar{D}, \qquad \Omega_c^T = D^c \times [0, T].$$

Consider the following initial–boundary value problem:

$$ {}_0^c D_t^\alpha u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega_c^T, \tag{1}$$

$$u|_{\Gamma_0} = g(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Gamma_0^T, \tag{2}$$

$$u|_{t=0} = u_0(\mathbf{x}), \quad \mathbf{x} \in D^c, \tag{3}$$

$$u(\mathbf{x}, t) \to 0, \quad \text{when } \|\mathbf{x}\| \to +\infty, \ t > 0, \tag{4}$$

where ${}_0^c D_t^\alpha$ $(0 < \alpha < 1)$ is the Caputo fractional derivative of order α defined by [1]

$$ {}_0^c D_t^\alpha u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x}, \lambda)}{\partial \lambda} (t - \lambda)^{-\alpha} d\lambda,$$

and $\mathbf{x} = (x_1, x_2)$ is the Cartesian coordinate in space, $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$. Δ is the Laplacian operator, i.e., $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. Functions $f(\mathbf{x}, t)$, $g(\mathbf{x}, t)$ and $u_0(\mathbf{x})$ are given smooth functions, and $f(\mathbf{x}, t)$ and $u_0(\mathbf{x})$ vanish outside a disc with the radius b, namely,

$$f(\mathbf{x}, t) = 0, \qquad u_0(\mathbf{x}) = 0,$$

for $\|\mathbf{x}\| \geqslant b$.

We introduce an artificial boundary

$$\Gamma = \left\{(\mathbf{x}, t) | \mathbf{x} \in D^c, \|\mathbf{x}\| = b, 0 \leqslant t \leqslant T\right\}, \tag{5}$$

where $a < b$.

Γ divides the domain Ω_c^T into two parts,

$$\begin{aligned} \Omega_i^T &= \left\{(\mathbf{x}, t) | \mathbf{x} \in D^c, \|\mathbf{x}\| < b, 0 \leqslant t \leqslant T\right\}, \\ \Omega_e^T &= \{(\mathbf{x}, t) | \, \|\mathbf{x}\| > b, 0 \leqslant t \leqslant T\}. \end{aligned} \tag{6}$$