Computers and Mathematics with Applications 67 (2014) 1673–1681

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Numerical simulation of a new two-dimensional variable-order fractional percolation equation in non-homogeneous porous media

S. Chen [a], F. Liu [b,*], K. Burrage [b,c]

[a] Department of Mathematics, Quanzhou Normal University, Quanzhou, Fujian, China

[b] School of Mathematical Sciences, Queensland University of Technology, Queensland 4001, Australia

[c] Department of Computer Science and OCISB, University of Oxford, OXI 3QD, UK

ARTICLE INFO

Article history:
Received 16 November 2013
Received in revised form 27 January 2014
Accepted 15 March 2014
Available online 3 April 2014

Keywords:
Fractional derivative of variable order
Fractional percolation equation
Variable coefficients
Implicit alternating direct method
Stability
Convergence

ABSTRACT

The inherent heterogeneities of many geophysical systems often give rise to fast and slow pathways to water and chemical movement. One approach to model solute transport through such media is by fractional diffusion equations with space–time dependent variable coefficients. Many physical processes appear to exhibit fractional-order behavior that may vary with time, or space, or space and time. The theory of pseudodifferential operators and equations has been used to deal with this situation. In this paper we use a fractional Darcy's law with variable order Riemann–Liouville fractional derivatives, this leads to a new variable-order fractional diffusion equation with variable coefficients.

In this paper we consider a new two-dimensional variable-order fractional percolation equation with variable coefficients. An alternating direct method for the two-dimensional variable-order fractional percolation equation is proposed. Stability and convergence of the implicit alternating direct method are discussed. Finally, some numerical results are given. The numerical results demonstrate the effectiveness of the methods. These techniques can be used to simulate three-dimensional variable-order fractional partial differential equations.

1. Introduction

Seepage flow problems are discussed in many research fields, such as seepage hydraulics, groundwater hydraulics, groundwater dynamics and fluid dynamics in porous media (see [1–3]). The traditional partial differential equations for single phase isothermal seepage flow under the hypotheses of continuity and Darcy's law can be written as

$$\frac{\partial}{\partial x}\left(A\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(B\frac{\partial p}{\partial y}\right)+h(x,y,t)=\frac{1}{\upsilon}\frac{\partial p}{\partial t}, \tag{1}$$

where $(x, y) \in \Omega$, A and B are the percolation coefficients along the x and y direction, respectively; p is the pressure; υ is velocity; $h = h(x, y, t)$ is the source term; and Ω denotes the percolation domain. He [3] proposed the following modified

* Corresponding author.
E-mail addresses: chenshiping65@163.com (S. Chen), f.liu@qut.edu.au (F. Liu), kevin.burrage@qut.edu.au (K. Burrage).

http://dx.doi.org/10.1016/j.camwa.2014.03.003

Darcy's law or generalized Darcy's law with Riemann–Liouville fractional derivatives:

$$q_x = A\frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}, \qquad q_y = B\frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}, \quad 0 < \alpha_1, \alpha_2 < 1. \tag{2}$$

Here, as usual, the Riemann–Liouville fractional derivative is defined as (see [4])

$$\frac{\partial^{\alpha} p}{\partial x^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x \frac{p(s,y,t)}{(x-s)^{\alpha}}ds, \tag{3}$$

where $0 < \alpha < 1$. Under the simplifying assumption of continuity of seepage flow, we have the following fractional differential equation:

$$\frac{\partial}{\partial x}\left(A\frac{\partial^{\alpha_1} p}{\partial x^{\alpha_1}}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial^{\alpha_2} p}{\partial y^{\alpha_2}}\right) + h(x,y,t) = \frac{1}{v}\frac{\partial p}{\partial t}. \tag{4}$$

Many physical processes appear to exhibit fractional-order behavior that may vary with time, or space, or space and time. The theory of pseudodifferential operators and equations has been used to deal with this situation (see [5–7]). In this paper we will consider a more generalized Darcy's law with variable order Riemann–Liouville fractional derivatives:

$$q_x = A\frac{\partial^{\alpha(x,y)} p}{\partial x^{\alpha(x,y)}}, \qquad q_y = B\frac{\partial^{\beta(x,y)} p}{\partial y^{\beta(x,y)}}, \quad 0 < \alpha(x,y), \beta(x,y) < 1. \tag{5}$$

Here, the variable order Riemann–Liouville fractional derivative is defined as (see [6,8])

$$\frac{\partial^{\alpha(x,y)} p}{\partial x^{\alpha(x,y)}} = \frac{1}{\Gamma(1-\alpha(x,y))}\left(\frac{\partial}{\partial \xi}\int_0^{\xi} \frac{p(s,y,t)}{(\xi-s)^{\alpha(x,y)}}ds\right)_{\xi=x} \tag{6}$$

where $0 < \alpha(x,y) < 1$. So the more general equation for seepage flow can be expressed as follows:

$$\frac{\partial}{\partial x}\left(A\frac{\partial^{\alpha(x,y)} p}{\partial x^{\alpha(x,y)}}\right) + \frac{\partial}{\partial y}\left(B\frac{\partial^{\beta(x,y)} p}{\partial y^{\beta(x,y)}}\right) + h(x,y,t) = \frac{1}{v}\frac{\partial p}{\partial t}. \tag{7}$$

The inherent heterogeneities of many geophysical systems often give rise to fast and slow pathways to water and chemical movement. One approach to model solute transport through such media is by fractional diffusion equations with a space–time dependent variable coefficient. Moreover, the equation in non-homogeneous porous media can be written as follows:

$$\frac{\partial}{\partial x}\left(A(x,y)\frac{\partial^{\alpha(x,y)} p}{\partial x^{\alpha(x,y)}}\right) + \frac{\partial}{\partial y}\left(B(x,y)\frac{\partial^{\beta(x,y)} p}{\partial y^{\beta(x,y)}}\right) + f(x,y,t) = \frac{\partial p}{\partial t}, \tag{8}$$

where $(x,y) \in \Omega,\ A(x,y) = vA,\ B(x,y) = vB,\ f(x,y,t) = vh(x,y,t)$. The above equation forms the focus of this paper and it is known as the two-dimensional variable order fractional percolation equation.

Many researchers have proposed various numerical methods to solve space or time fractional partial differential equations during the past decade (see [9–12]). Liu et al. [13] simulated Lévy motion with α-stable densities using a FADE. Meerschaert et al. [14] presented finite difference methods to solve the two-side space-fractional differential equations. Liu et al. [15] proposed an approximation of the Lévy–Feller advection–dispersion process by a random walk and finite difference method, and discussed its stability and convergence. Liu et al. [16] introduced numerical methods and analysis for a class of fractional advection–dispersion models. Liu et al. [17] proposed a new numerical simulation for two-dimensional Riesz space fractional diffusion equations with a nonlinear reaction term and discussed the stability and convergence of the new numerical method. Liu et al. [18] studied numerical methods for solving the multi-term time fractional wave equations. Meerschaert et al. [19] introduced stochastic models for fractional calculus. Ervin et al. [20] and Fix et al. [21] developed finite element methods for certain one-dimensional partial differential equations with constant coefficients in the fractional derivative terms. Recently, numerical methods for the fractional partial differential equations and the time–space fractional Fokker–Planck equations have been considered by some authors (see [22–25]). Lin et al. [26] develop modified alternating direction methods for solving a two-dimensional non-continuous seepage flow with fractional derivatives in uniform media. Lin et al. [6] and Zhuang et al. [8] develop numerical methods for solving the variable order fractional partial differential equations. But, numerical methods for the two-dimensional variable order fractional diffusion equation with variable coefficients are quite limited and difficult to construct. This motivates us to consider a computationally effective implicit alternating direct method for the two-dimensional variable order fractional diffusion equation with variable coefficients. The main contributions of this paper are based on existing two-dimensional fractional percolation equation with constant coefficients, we use a fractional Darcy's law with variable order Riemann–Liouville fractional derivatives, this leads to a new variable-order fractional diffusion equation with space–time dependent variable coefficients. A new implicit alternating direct method for the two-dimensional variable-order fractional percolation equation is proposed. Stability and convergence of the implicit alternating direct method are proved. Some numerical results are given. The numerical results demonstrate