Computers and Mathematics with Applications 67 (2014) 1682–1702

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Adaptively weighted numerical integration over arbitrary domains

Vaidyanathan Thiagarajan*, Vadim Shapiro

Spatial Automation Laboratory, University of Wisconsin - Madison, United States

ARTICLE INFO

Article history:
Received 4 October 2013
Received in revised form 10 January 2014
Accepted 1 March 2014
Available online 13 April 2014

Keywords:
Numerical integration
Quadrature
Cubature
Moment fitting equations
Shape sensitivity analysis
Meshfree analysis

ABSTRACT

In adaptively weighted numerical integration, for a given set of quadrature nodes, order and domain of integration, the quadrature weights are obtained by solving a system of suitable moment fitting equations in least square sense. The moments in the moment equations are approximated over a simplified domain that is homeomorphic to the original domain, and then are corrected for the deviation from the original domain using shape sensitivity analysis. Using divergence theorem, the moments reduce to integrals over the boundary of the simplified domain.

The proposed approach supports accurate and efficient computation of quadrature weights for integration of a priori unknown functions over arbitrary 2D and 3D solid domains. Experimental results (2D) indicate that adaptively weighted integration compares favorably with more traditional approaches. Because the adaptively weighted integration avoids excessive domain subdivision, it is useful in many applications and meshfree analysis in particular.

Published by Elsevier Ltd.

1. Introduction

1.1. Motivation

Numerical integration is a fundamental computation that is central to many problems in scientific and engineering computing, in large part because they require numerical integration of partial differential equations. For example, in structural analysis, numerical integration is used to assemble stiffness and mass matrices, as well as computing integral properties of physical fields such as average stresses and displacements. Formally, the problem is to numerically evaluate the integral:

$$\int_{\Omega} f(\mathbf{x})\, \mathrm{d}\Omega \tag{1}$$

where $f : \Omega \to \mathbb{R}$ is an integrable function defined over an arbitrary domain[1] $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) that is typically represented as either 2D or 3D solid model [1]. For the purposes of this paper, we will assume that the integrand function f is continuous.[2] If function f is known a priori, then the domain integral (1) can usually be reduced to a boundary integral

* Corresponding author. Tel.: +1 518 698 8380.
E-mail addresses: vthiagarajan@wisc.edu (V. Thiagarajan), vshapiro@engr.wisc.edu (V. Shapiro).

1 By 'arbitrary domain' we mean any 2D or 3D closed regular set [1] whose boundary is an orientable manifold.

2 For dealing with discontinuous functions refer to [2,3] and references therein.

http://dx.doi.org/10.1016/j.camwa.2014.03.001
0898-1221/Published by Elsevier Ltd.

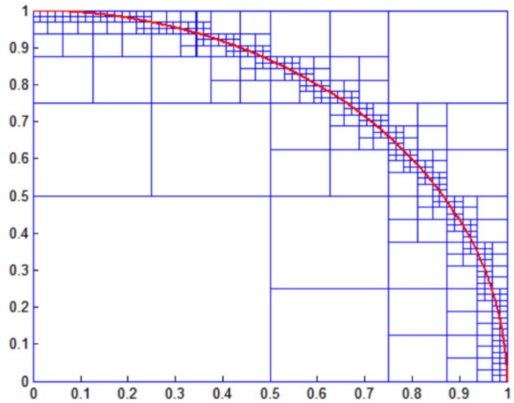

Fig. 1. Illustration of excessive boundary cell fragmentation in quadtree decomposition.

via application of the divergence theorem [4–8]. For the rest of the paper, we will assume that f is an arbitrary (continuous) function that is not known a priori.

Numerical integration over simple regular domains (interval, simplex, cube, and other ''cells'') is a classical widely studied topic that is relatively well understood [9,10]. Integration over more complex domains, such as those arising in most engineering applications, usually require that domain Ω is approximated by a union of simple cells, so that the integration can be carried out in a cell by cell fashion. In order to avoid integration errors, the union of cells must closely approximate the domain—either by cells that conform to the boundary $\partial\Omega$ or by non-conforming cells that are recursively decomposed to cover the original domain. The first approach leads to a difficult (theoretically and computationally) problem of meshing. The latter case, illustrated in Fig. 1, leads to excessive fragmentation near domain's boundary, resulting in significant computational cost and loss of accuracy. The goal of this paper is to devise an accurate and efficient method (called *Adaptively Weighted Numerical Integration* method) that avoids domain meshing and excessive fragmentation.

1.2. Overview and outline

Given an arbitrary domain Ω with a set of appropriately chosen quadrature nodes and order of integration, we compute the quadrature weights by solving a system of linear moment fitting equations [11–13] for an appropriate set of basis functions in the least square sense. Setting up the moment-fitting equations involves computing integrals of the basis functions (known as moments) over an arbitrary geometric domain. This task itself is non-trivial, because it either entails some kind of domain decomposition, or can be reduced to repeated boundary integration as was proposed in [14] for bivariate domains. Adaption of the latter approach avoids excessive domain fragmentation, but leads to a significant computational overhead, particularly in 3D.

We overcome this challenge by first computing the moments over a simpler domain Ω_0, usually a polygon/polyhedron, that is homeomorphic to and is a reasonable approximation (in a Hausdorff distance sense) of the original domain Ω. The computed moments are then corrected for the deviation from Ω based on first-order Shape Sensitivity Analysis (SSA) [15]. The moments over approximate domain are further reduced to boundary integrals by the application of divergence theorem [4–8]. The approximate moment fitting equations, thus obtained, can then be easily solved for quadrature weights that adapt to the original geometric domain—hence the name *Adaptively Weighted Numerical Integration (AW)*. The resulting weights are not exact, but are accurate enough to integrate functions over any arbitrary domain when they are well approximated by the chosen set of basis functions.

This integration scheme could be advantageously employed in cell decomposition methods, for example using quadtrees [16] or octrees [17], to reduce fragmentation. In Fig. 2(b), we show a simple comparison to demonstrate this. Here, the AW method proposed in this paper and Geometrically Adaptive (GA) integration method proposed by Luft et al. [18], are compared for integrating a 5th-degree polynomial ($f = x^2y^2 + x^2y^3 + x^3 + 100x + 10y + 2$) over a wavy domain (see Fig. 2(a)) using quadtree with 3-point integration rule in each of the cells. The analytical integral computed using MATLAB's symbolic toolbox is used as the reference for comparison. The graph in Fig. 2(b) clearly indicates that the proposed AW method is far more efficient than GA in terms of speed and memory, as it requires fewer quadtree subdivisions to achieve a desired accuracy. Additionally, the AW method is straightforward to implement and possesses a number of additional attractive features discussed in Section 5.2. The rest of the paper is organized as follows. Section 2 summarizes the relevant background material on geometrically adaptive methods of integration over piecewise linear and arbitrary domains. Section 3 summarizes the classical moment fitting approach and its relevance to the current formulation. The moment approximation for arbitrary domains is derived in Section 4. In Section 5, we outline the algorithm for adaptively weighted numerical integration (AW) and discuss its properties, including its running time. Section 6 describes experimental results and compares performance of AW method with other methods. The conclusion in Section 7 includes discussion of open issues that warrants further research.