Computers and Mathematics with Applications 67 (2014) 1722–1741

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An isogeometric analysis for elliptic homogenization problems

H. Nguyen-Xuan [a,*], T. Hoang [b], V.P. Nguyen [c]

[a] Vietnamese-German University, Le Lai Street, Hoa Phu Ward, Binh Duong New City, Binh Duong Province, Viet Nam
[b] IUSS—Istituto Universitario di Studi Superiori, Pavia, Italy
[c] School of Engineering, Cardiff University, Queen's Buildings, The Parade, Cardiff CF24 3AA, UK

ARTICLE INFO

Article history:
Received 16 August 2013
Received in revised form 19 December 2013
Accepted 5 January 2014
Available online 4 April 2014

Keywords:
Heterogeneous multiscale method
Homogenization
Isogeometric analysis (IGA)
NURBS
IGA-HMM

ABSTRACT

A novel and efficient approach which is based on the framework of isogeometric analysis for elliptic homogenization problems is proposed. These problems possess highly oscillating coefficients leading to extremely high computational expenses while using traditional finite element methods. The isogeometric analysis heterogeneous multiscale method (IGA-HMM) investigated in this paper is regarded as an alternative approach to the standard finite element heterogeneous multiscale method (FE-HMM) which is currently an effective framework to solve these problems. The method utilizes non-uniform rational *B*-splines (NURBS) in both macro and micro levels instead of standard Lagrange basis. Besides the ability to describe exactly the geometry, it tremendously facilitates high-order macroscopic/microscopic discretizations thanks to the flexibility of refinement and degree elevation with an arbitrary continuity level provided by NURBS basis functions. Furthermore, the nearly optimal quadrature rule for IGA (Auricchio et al., 2012) introduced recently is utilized to reduce significantly the number of micro problems, which is the main factor contributing into the computational cost in heterogeneous multiscale method. A priori error estimates of the discretization error coming from macro and micro meshes and optimal micro refinement strategies for macro/micro NURBS basis functions of arbitrary orders are derived. An efficient coupling between degrees of macro and micro basis functions is introduced. Numerical results show the excellent performance of the proposed method.

1. Introduction

Homogenization is a branch of mathematics and engineering which studies partial differential equations (PDEs) with rapidly oscillating coefficients. These kinds of equations describe various processes in heterogeneous materials with rapidly oscillating micro structures, such as composite and perforated materials, and thus play an important role in physics, engineering and modern technologies. The aim of homogenization is to "average out" the heterogeneities at microscale and describe the effective properties at macro-scale of such phenomena. In other words, we want to know the behaviors of the systems as homogeneous ones. In analytical approaches, such as in [1,2], homogenized equations are derived. However, the

* Corresponding author.
E-mail address: hung.nx@vgu.edu.vn (H. Nguyen-Xuan).

0898-1221/$ – see front matter © 2014 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.camwa.2014.01.001

coefficients of these equations are only computed explicitly in some special cases, such as when the medium follows some periodic assumptions, and not explicitly available in general. Furthermore, full computations with complex scale interactions of the heterogeneous system are very ineffective due to high computational cost. Thus, to solve these problems, advanced computational technologies have been developed.

Literature reviews and recent developments on various multiscale approaches can be found in [3–7]. In this paper, we focus on the heterogeneous multiscale method (HMM), which was proposed in [8]. Reviews on HMM are presented in [9–11,3]. This method provides a general framework which allows one to develop various approaches to homogenization problems. The simplest one is the finite element heterogeneous multiscale method (FE-HMM), which uses standard finite elements such as simplicial or quadrilateral ones in both macroscopic and microscopic level. Solving the so-called micro problems (with a suitable set up) in sampling domains around traditional Gauss integration points at macro level allows one to approximate the missing effective information for the macro solver.

The standard finite element method (FEM), albeit very popular in various fields of sciences and engineering, still has some shortcomings which affect the efficiency of the FE-HMM. Firstly, the discretized geometry through mesh generation is required. This process often results in geometrical errors even with the higher-order FEM. Also, the communication between the geometry model and the mesh program during the analysis process is always needed and this constitutes the large part of the overall process [12], especially for industrial problems. Secondly, lower-order formulations, such as FE-HMM based on the four-node quadrilateral element (Q4), often require extremely fine meshes to produce approximate solutions with a desired accuracy for complicated problems. This prevents multiscale analyses from being run on personal computers. Thirdly, high-order formulations still put some restrictions on the element topologies (for example, the connection of different type of corner, center, or internal nodes) and only possess C^0 continuity. These disadvantages lead to an increase in the number of micro coupling problems and thus increase the overall computational cost. Hence, there is a need to consider alternative methods to tackle these issues.

Among advanced numerical methods, the so-called isogeometric analysis (IGA), where NURBS are used as basis functions, emerges as the most potential candidate. The isogeometric analysis was first proposed in [12] and now has attracted the attention of academic as well as engineering community all over the world. The IGA provides a framework in which the gap between Computer Aided Design (CAD) and Finite Element Analysis (FEA) may be reduced. This is achieved in IGA by employing the same basis functions to describe both the geometry of the domain of interest and the field variables. While the standard FEM uses basis functions which are based on Lagrange polynomials, isogeometric approach utilizes more general basis functions such as *B*-splines and NURBS that are commonly used in CAD geometry. The exact geometry is therefore maintained at the coarsest level of discretization and re-meshing is seamlessly performed on this coarse level without any further communication with CAD geometry. Furthermore, *B*-splines (or NURBS) provide a flexible way to make refinement, de-refinement, and degree elevation [12]. They allow us to achieve easily the smoothness of arbitrary continuity in comparison with the C^0 continuity provided by the traditional FEM. For a reference on IGA, we recommend the excellent book [13] and we refer to the NURBS book [14] for a geometric description.

In this work, we introduce a new approach: a so-called isogeometric analysis heterogeneous multiscale method (IGA-HMM) which utilizes NURBS as basis functions for both exact geometric representation and analysis. The NURBS are used as basis functions for both macro and micro element spaces, where the FE-HMM only employs the standard FEM basis. This tremendously facilitate high-order macroscopic/microscopic discretizations by a flexibility of refinements and degree elevations with an arbitrary continuity of basis functions. As will be demonstrated later in the numerical examples section, elliptic homogenization problems can be solved, by using the proposed IGA-HMM, effectively on a personal computer which is not the case for the FE-HMM with (bi)linear basis functions if high accuracies are needed.

The FE-HMM with bi(linear) elements often requires very fine macro meshes (thus a high number of micro problems) that are only supported by powerful computers. It is obvious that one can use high order Lagrange elements in the framework of FE-HMM to achieve the same goal. However, high order Lagrange elements cannot be constructed as straightforwardly as NURBS and more importantly they are only C^0 continuity whereas NURBS elements are C^{p-1} (p is the NURBS order) by definition. Therefore, IGA-HMM is able to solve high order homogenization problems such as plate/shell homogenization problems. Furthermore, the high order continuity of NURBS also allows one to develop new quadrature rules for IGA, in which the number of quadrature points is reduced to about a half on each dimensional space [15,16]. This helps the IGA-HMM reduce the large number of micro problems, and brings out an outstanding advantage of the IGA-HMM in comparison with the standard FE-HMM. In addition, one can consider our IGA-HMM as an efficient high order NURBS-based FE-HMM. We refer to [17,18] for a related work on high order FE-HMM which is however restricted to cubic polynomial basis functions. A priori error estimates of the discretization error coming from macro and micro meshes are provided. Optimal micro refinement strategies for macro and micro NURBS basis functions of arbitrary orders are presented and an optimal value for the micro NURBS order is thus obtained as a function of the macro NURBS basis order. We have found that C^0, not C^{p-1}, high order NURBS should be used at the microscale. It should be mentioned that the proposed IGA-HMM is very similar to the standard FE-HMM, as compared to other non-standard high order FEM-HMM [17,18], this means that the implementation is simple, and any existing coding framework of FE-HMM e.g. the one given in [19] can be readily re-used.

The paper is arranged as follows: a brief introduction to the *B*-splines, NURBS and IGA is given in Section 2. Section 3 describes the isogeometric analysis heterogeneous multiscale method. Two numerical examples are provided in Sections 4 and 5 closes the paper with some concluding remarks.