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A regularized optimization method for identifying the space-dependent source and the initial value simultaneously in a parabolic equation

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ABSTRACT

In this paper, a regularized optimization method is proposed for identifying the space-dependent source and the initial value simultaneously in an inverse parabolic equation problem from two over-specified measurements at different instants of time. The solvability of the direct problem is presented and then the inverse problem is formulated into a regularized optimization problem for the stable identification of both the source term and the initial value. Based on a sequence of well-posed direct problems solved by the finite element method, a numerical scheme formulated into a linear system is proposed to implement the regularized optimization problem. Numerical results of three examples show that the proposed method is efficient and robust with respect to data noise.

1. Introduction

In this study, we consider an inverse problem of a parabolic equation which aims to identify the space-dependent source term and the initial value simultaneously.

Let Ω be a bounded domain possessing piecewise-smooth boundaries in the Euclidean space $\mathbf{R}^n$, $n \geq 1$. $x = (x_1, x_2, \ldots, x_n)$ denotes an arbitrary point in Ω, and $\partial\Omega$ is used for the boundary of the domain Ω. Let us denote by Q_T a cylinder $\Omega \times (0, T)$ consisting of all points $(x, t) \in \mathbf{R}^{n+1}$ with $x \in \Omega$ and $t \in (0, T)$. The concentration $u(x, t)$ and the initial value $\varphi(x)$ satisfy the parabolic equation

$$u_t(x, t) = (Lu)(x, t) + f(x), \quad (x, t) \in \Omega \times (0, T), \tag{1.1}$$

the initial condition

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \tag{1.2}$$

and the homogeneous boundary condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \tag{1.3}$$

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where the source $f(x)$ and the initial value $\varphi(x)$ are two unknown functions to be identified from the following two over-specified measurements

$$u^\delta(x, T_1) = \Psi_1^\delta(x), \qquad u^\delta(x, T_2) = \Psi_2^\delta(x), \quad x \in \Omega. \tag{1.4}$$

Here, δ is the noise level satisfying $\|u(x, T_i) - u^\delta(x, T_i)\|_{L^2(\Omega)} \le \delta$ for $i = 1, 2$, where $u(x, T_i)$ is the exact value at the moment T_i. L is a linear elliptic operator of second order, i.e.,

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

Moreover, the operator L is supposed to be uniformly elliptic, which means that $a_{ij}(x) = a_{ji}(x)$ and

$$0 < \nu \sum_{i=1}^{n} \zeta_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \le \mu \sum_{i=1}^{n} \zeta_i^2 \tag{1.5}$$

with positive constants ν and μ, and arbitrary point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{R}^n$. Considering some practical characters in engineering applications, we confine the coefficients of the operator L to satisfying that

$$a_{ij}(x), b_i(x), c(x) \in C(\bar{\Omega}), \qquad \frac{\partial a_{ij}(x)}{\partial x_k} \in C(\bar{\Omega}), \quad k = 1, 2, \ldots, n. \tag{1.6}$$

Only numerical reconstruction of an unknown space-dependent source in parabolic equations has been studied widely in recent years, for example, see [1–9]. In [7,8], two iterative methods were proposed for finding the spacewise dependent source: one is an iterative algorithm based on a sequence of well-posed direct problems; the other is a variational conjugate gradient-type iterative algorithm which also need to solve a sequence of well-posed direct problems at each iteration. The paper [9] is devoted to identify an unknown heat source depending simultaneously on both space and time variables that is transformed into an optimization problem. Moreover, an inverse point source problem has been considered by the authors in [10]. On the other hand, the inverse problem of only determining the initial value in parabolic equations, i.e., the backward parabolic equation problem, has been investigated extensively, see [11–16] and references therein.

The inverse problem of identifying both the source term $f(x)$ and the initial value $\varphi(x)$ from the two measurements $\Psi_1^\delta(x)$ and $\Psi_2^\delta(x)$ was first studied in [17], where the authors presented an iterative regularization algorithm and used the boundary element method to solve the direct problem at each iteration. Thenceforth, this inverse problem was investigated in [18] with respect to the standard heat conduction equation, where the authors transformed it into a homogeneous backward heat conduction problem and a Dirichlet boundary value problem for Poisson equation. The backward heat conduction problem is an ill-posed problem solved by a regularized method based on the method of fundamental solutions; the Dirichlet boundary problem is a well-posed problem solved by the finite element method. It is also worth noting that the simultaneous reconstruction of the initial value and heat radiative coefficient was studied in [19], where the inverse problem was formulated into a nonlinear optimization problem by Tikhonov regularization with the regularized terms being the L_2-norms of gradients. To implement this inverse problem successfully, the authors discretized the continuous nonlinear optimization problem by the finite element method. Subsequently, the optimization technique was developed by the authors in [20] to recover heat fluxes on an inner boundary of a heat conductive system from the measurement of temperature in a small subregion. Similar optimization techniques can also be found in [13,21,22] and references therein.

In this paper, we proposed a regularized optimization method to identify the space-dependent source term and the initial value simultaneously. The inverse problem is firstly formulated into a regularized optimization problem. Then, based on a sequence of well-posed direct problems solved by the finite element method, the optimization problem is reduced to a system of linear algebraic equations which can be solved directly for obtained approximate solutions of both the source term and the initial value. This paper is organized as follows. In Section 2, some properties of the direct problem are given. In Section 3, the inverse problem is formulated as a regularized optimization problem with its approximation by the finite element method. In Section 4, the implementations of the regularized optimization method are proposed in detail. In Section 5, some numerical results are given to illustrate the efficiency and stability with respect to data noise.

2. Solvability of the direct problem

2.1. functional spaces

In this subsection, we give some preliminary functional spaces [23]. The space $L_2(\Omega)$ is a Banach space consisting of all square integrable functions on the domain Ω with the norm

$$\|u\|_{2,\Omega} = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}$$