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Radial basis function method for a multidimensional linear elliptic equation with nonlocal boundary conditions

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ABSTRACT

The development of numerical methods for the solution of partial differential equations (PDEs) with nonlocal boundary conditions is important, since such type of problems arise as mathematical models of various real-world processes. We use radial basis function (RBF) collocation technique for the solution of a multidimensional linear elliptic equation with classical Dirichlet boundary condition and nonlocal integral conditions. RBF-based meshless methods are easily implemented and efficient, especially for multidimensional problems formulated on complexly shaped domains. In this paper, properties of the method are investigated by studying two- and three-dimensional test examples with manufactured solutions. We analyze the influence of the RBF shape parameter and the distribution of the nodes on the accuracy of the method as well as the influence of nonlocal conditions on the conditioning of the collocation matrix.

1. Introduction

Elliptic partial differential equations (PDEs) have important applications in various areas of mathematics and physics. For example, the Laplace equation arises in modeling various kinds of conservative physical systems in equilibrium, and the Poisson equation has many applications in electrostatics, mechanical engineering and theoretical physics. The Helmholtz equation appears in mathematical models related to steady-state oscillations in mechanics, acoustics, electromagnetics etc. In mathematical modeling, elliptic PDEs are used together with boundary conditions specifying the solution on the boundary of the domain. Dirichlet, Neumann and Cauchy conditions are examples of classical boundary conditions. In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions usually are identified as *nonlocal (boundary) conditions* and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. For example, we can mention problems in thermoelasticity [1], thermodynamics [2], hydrodynamics [3] or plasma physics [4].

We consider a multidimensional linear elliptic equation

$$\mathcal{L}[u] := -\frac{\partial^2 u}{\partial y^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u = f \quad \text{in } \Omega = \omega \times (0, 1), \tag{1}$$

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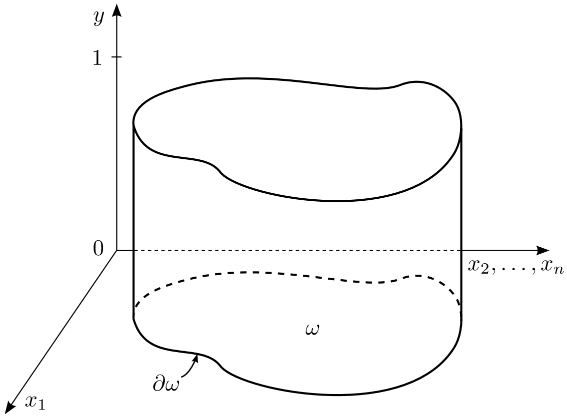

Fig. 1. Sketch of the domain Ω.

subject to classical Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\omega \times (0, 1), \tag{2}$$

and nonlocal integral conditions

$$\int_0^{\xi_1} u(\mathbf{x}, y)\mathrm{d}y = 0, \qquad \int_{\xi_2}^1 u(\mathbf{x}, y)\mathrm{d}y = 0, \quad \mathbf{x} \in \omega, \tag{3}$$

where $\omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with Lipschitz boundary $\partial\omega$, $0 < \xi_1 < \xi_2 < 1$ and $f : \Omega \to \mathbb{R}$, $a_{ij}, a_0 : \omega \to \mathbb{R}$ are given functions from suitable spaces. The sketch of the cylindrical domain Ω is given in Fig. 1.

The existence and uniqueness results for this problem have been obtained by Avalishvili et al. [5,6].

Theorem *([5,6]). If $f \in L^2(\Omega)$, the coefficients $a_{ij}, a_0 \in L^\infty(\omega)$ and satisfy the conditions*

$$a_0(\mathbf{x}) \geqslant 0, \qquad a_{ij} = a_{ji},$$
$$\sum_{i,j=1}^{n} a_{ij}\xi_j\xi_i \geqslant c_a \sum_{i=1}^{n} |\xi_i|^2, \quad \forall \xi_i \in \mathbb{R},\ i, j = 1, 2, \ldots, n,$$

where c_a is a positive constant (see [5,6]), then the problem (1)–(3) has a unique solution.

PDEs with nonlocal conditions and numerical methods for their solution are popular objectives of research. The finite difference schemes for two-dimensional elliptic problems with nonlocal boundary conditions have been considered in papers [3,7–11]. In papers [12,13], the solvability of the finite difference schemes for similar problems has been proved and the discretization error estimates have been obtained. Some error estimates on the finite element approximation for two-dimensional elliptic problem with nonlocal boundary conditions are given in paper [14]. In paper [15], a high order composite scheme for the second order elliptic problem is presented and a fast algorithm for its solution is designed. A new constructive method for the solution of the two-dimensional Poisson equation with Bitsadze–Samarskii nonlocal boundary condition has been proposed [16]. Various finite difference schemes for the solution of multidimensional nonlocal elliptic boundary value problems have been studied in paper [17]. In papers [18,19], stable finite difference schemes for the solution of multidimensional elliptic equations with multipoint nonlocal boundary conditions have been proposed and analyzed. However, in papers mentioned above, problems were formulated on regular domains and numerical experiments were performed only with two-dimensional test examples.

Traditional numerical techniques for the solution of PDEs (such as finite differences, finite volumes, finite elements etc.), are based on the domain discretization using meshes. For multidimensional problems formulated on complexly shaped domains, mesh generation can be very time consuming. Recently, meshless methods for the numerical solution of PDEs became popular between scientists and engineers [20,21]. Such type of methods do not require mesh generation or remeshing.

The radial basis function (RBF) collocation methods is one class of meshless methods. RBFs successfully can be used both for the interpolation of scattered multidimensional data and for the numerical solution of PDEs [22–24]. The superiority of the RBF technique against the finite difference method [25], or finite difference and dual reciprocity methods [26], or finite difference and pseudospectral methods [27] already has been demonstrated. In paper [28] it has been demonstrated that the RBF collocation method for the solution of elliptic equations with classical boundary conditions is more accurate than the finite element method with the same mesh. Papers [27,29–43] are dedicated especially to questions related to the solution of various elliptic problems with classical boundary conditions using RBF-based meshless methods.