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Finding the number of roots of a polynomial in a plane region using the winding number

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ABSTRACT

We describe a method that computes the number of roots of a polynomial f inside a region bounded by the curve Γ, with an analysis of its computational cost. It is based on the number of roots being the same as the winding number of $f(\Gamma)$. While the usual methods for computing the winding number involve numerical integration, in this paper we use a geometrical construction. We show its correctness without referring to global information about f (like its Lipschitz constant on Γ). The analysis of its cost is based on the distance from the roots to Γ, expressed using a condition number suitably defined. The method can be used in a divide-and-conquer root-finding algorithm.

1. Introduction

Let us consider, for expository purposes, the bisection method for zero finding of real continuous functions. It is based on Bolzano's theorem. This theorem says that, if a continuous function $f : [a, b] \to \mathbb{R}$ in an interval $[a, b]$ of the real straight takes opposed signs at its ends (that is, $f(a) \cdot f(b) < 0$), then it has at least one root in the interval. The bisection method consists of using this fact recursively to build a succession of nested (and hence progressively smaller) intervals containing a real root. These are the prototype components of a divide-and-conquer root-finding method: an *inclusion test*, to decide whether there is any root in a piece (either of the line or of the plane); and a *recursive procedure* to get progressively more precise localizations of the required roots. In this paper we focus on an inclusion test (see [1] for a recursive procedure).

A result that is often used as an inclusion test is the principle of the argument from Complex Analysis, which equates the number of zeros of a polynomial f within a region of border Γ with the winding number of the curve $\Delta = f(\Gamma)$. The winding number is the number of twists of Δ around the origin. We have developed [2] an efficient procedure to compute the winding number for general curves Δ, that is, curves not necessarily of the form $f(\Gamma)$. In this work we expose an inclusion test for polynomials based on it, the IPSR, in Fig. 8.

Our inclusion test is not of algebraic type, as those using the Descartes test [3] or Sturm sequences [4], but of analytical type. The work [5], which describes a method for isolating roots similar to ours, also uses an analytical test, called the 8-point test, which is essentially the computation of the winding number of certain discs. They also proposed a recursive procedure for the decomposition of the search area, tightly coupled with the 8-point test. The method for winding number computation that we discuss below is more general than the 8-point test, so the recursive procedure can be exposed independently, in a modular way, as we do in [1]. Another difference between the referred work and ours is that [5] develops an analysis of

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bit complexity, suitable for an implementation in exact arithmetic, while we carry out a floating point complexity analysis. Our implementation can be tested in the web page http://gim.unex.es/contour/.

We now describe two particular contributions of the paper: the analysis of the computational cost of the IPRS without using global information about polynomial f, and the use of a condition number to avoid ill-conditioned curves. The zero-finding methods (for polynomials or more general functions) based on the winding number compute it either by numerical integration (following [6]) or by geometrical discretization of the curve (following [7]). The survey in the introduction of [8] compares the respective advantages. Despite the greater computational cost of the integration methods, they have been preferred in practice because of the form of the error bounds. These error bounds depend only on the value of the function at certain points, so the cost of reaching a predefined accuracy is predictable. In contrast, the error bounds for discretization methods have been based on some form of global information (such as the maximum of the derivative of the function in a region, or its Lipschitz constant), which is often not available. Our IPSR method, valid for polynomials f, uses discretization, but we prove its correctness without global information, using only the values of f or f' in a finite set of complex points.

The ill-conditioned curves (that is, the curves passing over a root of f, or with a root closer than some threshold) are another disadvantage of the discretization methods, because the computational cost in these curves is very high [9]. The IPSR solves this issue with a condition number, similar to the condition number used in Numerical Lineal Algebra [10] for the computation of ill-conditioned eigenvalues. In the ill-conditioned curves the winding number is not computed, but it is found instead a bound in the condition number that allows us to locate a root closer than some threshold to the curve. This fact can be subsequently used in a divide-and-conquer method, as described for example in [1].

This condition number, which controls the ill-conditioned curves, is related to the notion of (δ, ε)-completeness, exposed in [11]. This approach arises from the theory of Exact Geometric Computation [12] to deal with data types that require unbounded amount of resources, for example a real number of arbitrary precision. In practice, we can reach only finite approximations of these data types, leading to problems such as zero test: we cannot be sure that a real x is zero. At best we can test whether it is greater than (or less than) zero. But to ensure equality, we need access to the infinite representation of x. It is said that, on a geometric data type, we can only apply soft zero tests, such as $(x > 0)$?, but not strict tests, such as $(x \geq 0)$? or $(x = 0)$? An algorithm that works with soft tests usually requires certain bounds in their inputs to produce correct outputs [11]. For example, it requires that the absolute value of x is above a threshold. Certainly it is desirable to have a complete algorithm, that is, one without such restrictions on the input. Even using soft tests it is often possible to develop (δ, ε)-complete algorithms, meaning that for each threshold δ bounding the input there is a bound ε in the error of the output. With this terminology, in [2] we give a (δ, ε)-complete algorithm for computing the winding number of a plane curve, and in this paper we give a (δ, ε)-complete algorithm for the computation of the number of roots of a polynomial surrounded by a plane curve.

For our algorithm, the condition number is the sum of the reciprocal of the distances to the roots of the polynomial (see Section 3). It is remarkable that the proof of the (δ, ε)-completeness, which gives us the effectiveness of the algorithm as well as its cost, is related to this condition number, which appears in a similar expression in the complexity analysis of [5].

This paper is structured as follows. Section 2 summarizes the procedure of winding number computation for a general curve Δ, and the results of [2] about its applicability and cost, mainly in terms of the distance from the origin to Δ. In Section 3, we describe the IPRS, adapting these results to specific curves of the form $\Delta = f(\Gamma)$, f being a polynomial and Γ a curve surrounding a plane region. The cost bound in this case is expressed in terms of the distance to every root from the curve Γ. We show the correctness of the IPRS for curves with low condition number, and otherwise that there is a root below some threshold distance from the curve.

2. Computing the winding number

Let us consider a polynomial f and the function that it defines: $f : \mathbb{C} \to \mathbb{C}$. As notation, we call z a complex of the domain of f and w a complex of its codomain. We also consider closed curves defined parametrically, that is, as mappings of an interval to the complex plane, $\Gamma : [a, b] \to \mathbb{C}$, with $\Gamma(a) = \Gamma(b)$ in the domain and Δ in the codomain.

The principle of the argument [7] of Complex Analysis states that the number of zeros N (counted with multiplicity) of an analytic function, in particular of the polynomial $f : \mathbb{C} \to \mathbb{C}$, $w = f(z)$, inside a region with border defined by the curve Γ, is equal to this contour integral:

$$N = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)}\, dz.$$

Moreover, the *winding number*, or *index*, $\mathrm{Ind}(\Delta)$, of a curve $\Delta : [a, b] \to \mathbb{C}$, is the number of complete rotations, in the counterclockwise sense, of the curve around the point $(0, 0)$. See Fig. 1. The Cauchy formula equates the integral $\frac{1}{2\pi i} \oint_{\Delta} \frac{1}{w}\, dw$ to the winding number of Δ.

If the curve Δ is the transformation of Γ by the polynomial f, that is, if $\Delta = f(\Gamma)$, then these two contour integrals are equal, by the variable change $w = f(z)$. As consequence, the number of roots of f inside Γ is $\mathrm{Ind}(f(\Gamma))$. See Fig. 2. This can be used as an inclusion test to decide if a given region of the plane has any root.

It should be noted that the winding number of the curve Δ is not defined if Δ crosses over the origin $(0, 0)$. In that case, Δ is called a *singular curve*, since the integral $\int_{\Delta} \frac{1}{w} dw$ does not exist. If $\Delta = f(\Gamma)$, this is equivalent to Γ crossing over a root of f. In that case, Γ is called a singular curve with respect to f, and the principle of argument is not applicable.