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An efficient fifth order method for solving systems of nonlinear equations

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ABSTRACT

In this paper, we present a three-step iterative method of convergence order five for solving systems of nonlinear equations. The methodology is based on the two-step Homeier's method with cubic convergence (Homeier, 2004). Computational efficiency in its general form is discussed and a comparison between the efficiency of proposed technique and existing ones is made. The performance is tested through numerical examples. Moreover, theoretical results concerning order of convergence and computational efficiency are verified in the examples. It is shown that the present method has an edge over existing methods, particularly when applied to large systems of equations.

1. Introduction

The construction of iterative methods for approximating the solution of systems of nonlinear equations is an important and interesting task in numerical analysis and applied scientific branches. With the advancement of computers, the problem of solving systems of nonlinear equations by numerical methods has gained more importance than before. This problem can be precisely stated as to find a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ such that $F(\alpha) = 0$, where $F(x) : D \subseteq R^n \to R^n$ is the given nonlinear system, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^t$ and $x = (x_1, x_2, \ldots, x_n)^t$. One of the basic procedures for solving nonlinear equations is the classical Newton's method [1,2] which converges quadratically under the conditions that the function F is continuously differentiable and a good initial approximation $x^{(0)}$ is given. It is defined by

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \quad k = 0, 1, 2, \ldots,$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$. It is straightforward to see that this method requires the evaluations of one function, one first derivative and one matrix inversion per iteration.

In order to improve the order of convergence of Newton's method, many modifications have been proposed in the literature; for example, see [3–14] and references therein. In particular, Homeier [8] has developed a two-step cubically convergent method, which is given by

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} &= x^{(k)} - F'(y^{(k)})^{-1}F(x^{(k)}). \end{aligned} \tag{1}$$

Per iteration, this method requires the evaluations of one function, two first order derivatives and two matrix inversions.

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A well-known fact in numerical analysis is that the construction of higher order iterative methods is a futile exercise unless they have low computational cost. Thus, the main goal and motivation in developing iterative methods is to achieve as high as possible convergence order requiring as small as possible the evaluations of functions, derivatives and inverse operators. With this aim we here propose a three-step method of fifth order convergence based on Homeier's scheme (1). The scheme of present contribution is as simple as the method (1) but with an additional advantage that it possesses high computational efficiency, particularly for large systems.

This paper is outlined as follows. In Section 2, the fifth order scheme is developed and its convergence analysis is studied. In Section 3, the computational efficiency of new method is discussed and is compared with Homeier's (1) and other well-known existing methods. Various numerical examples are considered in Section 4 to show the consistent convergence behavior of the method and to verify the theoretical results. Section 5 contains the concluding remarks.

2. The method and its convergence

Consider Homeier's method (1), which is now defined as

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F(x^{(k)}), \\ z^{(k)} &= x^{(k)} - F'(y^{(k)})^{-1}F(x^{(k)}). \end{aligned} \tag{2}$$

In what follows, we construct the method to obtain the approximation $x^{(k+1)}$ to a solution of $F(x) = 0$ by considering the scheme in the following way:

$$\begin{aligned} y^{(k)} &= x^{(k)} - \frac{1}{2}F'(x^{(k)})^{-1}F(x^{(k)}), \\ z^{(k)} &= x^{(k)} - F'(y^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} &= z^{(k)} - [aF'(y^{(k)})^{-1} + bF'(x^{(k)})^{-1}]F(z^{(k)}) \end{aligned} \tag{3}$$

where a and b are some parameters. In order to explore the convergence property of (3), we recall the following result of Taylor's expansion on vector functions (see [1]).

Lemma 1. *Let $F : D \subseteq R^n \to R^n$ be a p time Fréchet differentiable in a convex set $D \subseteq R^n$ and then for any $x, h \in R^n$, the following expression holds:*

$$F(x+h) = F(x) + F'(x)h + \frac{1}{2!}F''(x)h^2 + \frac{1}{3!}F'''(x)h^3 + \cdots + \frac{1}{p!}F^{(p-1)}(x)h^{p-1} + R_p, \tag{4}$$

where

$$\|R_p\| \leqslant \frac{1}{p!}\sup_{0\leqslant t\leqslant 1}\|F^{(p)}(x+th)\|\,\|h\|^p \quad \textit{and} \quad h^p = (h, h, \cdot \overset{p}{\cdot\cdot}, h).$$

We are in a condition to prove the following theorem:

Theorem 1. *Let the function $F : D \subseteq R^n \to R^n$ be sufficiently differentiable in a convex set D containing the zero α of $F(x)$. Suppose that $F'(x)$ is continuous and nonsingular in α. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ ($x^{(0)} \in D$) obtained by using the method (3) converges to α with convergence order five provided $a = 2$ and $b = -1$.*

Proof. Taylor's expansion (4) for $F(x)$ about $x^{(k)}$ is

$$F(x) = F(x^{(k)}) + F'(x^{(k)})(x - x^{(k)}) + \frac{1}{2!}F''(x^{(k)})(x - x^{(k)})^2 + \frac{1}{3!}F'''(x^{(k)})(x - x^{(k)})^3 + O(\|x - x^{(k)}\|^4). \tag{5}$$

Let $e^{(k)} = x^{(k)} - \alpha$. Then setting $x = \alpha$ and using $F(\alpha) = 0$ in (5), we obtain

$$F(x^{(k)}) = F'(\alpha)[e^{(k)} + A_2(e^{(k)})^2 + A_3(e^{(k)})^3 + O((e^{(k)})^4)], \tag{6}$$

where $A_i = \frac{1}{i!}\Gamma F^{(i)}(\alpha) \in L_i(R^n, R^n)$, $\Gamma = F'(\alpha)^{-1}$ and $(e^{(k)})^i = (e^{(k)}, e^{(k)}, \cdot \overset{i}{\cdot\cdot}, e^{(k)}),\ e^{(k)} \in R^n$.
Also,

$$F'(x^{(k)}) = F'(\alpha)[I + 2A_2e^{(k)} + 3A_3(e^{(k)})^2 + O((e^{(k)})^3)]. \tag{7}$$

Then,

$$F'(x^{(k)})^{-1} = [D(e^{(k)})^{-1} + O((e^{(k)})^3)]\Gamma,$$

where $D(e^{(k)}) = I + 2A_2e^{(k)} + 3A_3(e^{(k)})^2$.