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Uniformly convergent additive finite difference schemes for singularly perturbed parabolic reaction–diffusion systems

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ABSTRACT

In this paper 1D parabolic systems of two singularly perturbed equations of reaction–diffusion type are examined. For the time discretization we consider two additive (or splitting) schemes defined on a uniform mesh and for the space discretization we use the classical central difference approximation defined on a Shishkin mesh. The uniform convergence of both the semidiscrete and the fully discrete problems is proved. The additive schemes are used to solve a test problem, and the results obtained with these schemes and the standard discretization using the backward Euler method are compared. Also, numerical results are presented in the case of systems of three equations. All the numerical results show the advantage in computational cost of the additive schemes compared to the standard discretization.

1. Introduction

In this paper we are interested in the numerical approximation of parabolic singularly perturbed reaction–diffusion systems of type

$$\begin{cases} L_\varepsilon \mathbf{u} \equiv \dfrac{\partial \mathbf{u}}{\partial t}(x,t) + \mathcal{L}_{x,\varepsilon}\mathbf{u}(x,t) = \mathbf{f}(x,t), & (x,t) \in Q = \Omega \times (0,T], \\ \mathbf{u}(0,t) = \mathbf{0}, \quad \mathbf{u}(1,t) = \mathbf{0}, \quad \forall\, t \in [0,T], & \mathbf{u}(x,0) = \mathbf{0}, \quad \forall\, x \in \bar{\Omega}, \end{cases} \tag{1}$$

where $\Omega = (0,1)$ and the spatial differential operator is defined by

$$\mathcal{L}_{x,\varepsilon} \equiv \mathcal{D}_\varepsilon \frac{\partial^2}{\partial x^2} + \mathcal{A}, \quad \mathcal{D}_\varepsilon = \begin{pmatrix} -\varepsilon_1 & \\ & -\varepsilon_2 \end{pmatrix}, \mathcal{A} = \begin{pmatrix} a_{11}(x,t) & a_{12}(x,t) \\ a_{21}(x,t) & a_{22}(x,t) \end{pmatrix}. \tag{2}$$

We denote by $\Gamma_0 = \{(x,0) \mid x \in \Omega\}$, $\Gamma_1 = \{(x,t) \mid x = 0,1,\ t \in [0,T]\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$ the vectorial singular perturbation parameter and we assume that $0 < \varepsilon_1 \le \varepsilon_2 \le 1$. The components of the right hand side function $\mathbf{f}(x,t) = (f_1(x,t), f_2(x,t))^T$ and the reaction matrix $\mathcal{A}$ are assumed to be sufficiently smooth functions. Also, we suppose that the following conditions on $\mathcal{A}$ are satisfied:

$$a_{i1} + a_{i2} \ge \alpha \ge 0, \qquad a_{ii} > 0, \quad i = 1,2, \tag{3}$$
$$a_{ij} \le 0, \quad \text{if } i \ne j. \tag{4}$$

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Note that if (3) is not satisfied, we can consider the transformation $\mathbf{v}(x,t)=\mathbf{u}(x,t)e^{-\alpha_0 t}$ with $\alpha_0>0$ sufficiently large, and therefore condition (3) will hold in the new problem. We also assume the compatibility conditions

$$\frac{\partial^{i+k}\mathbf{f}}{\partial x^i\partial t^k}(0,0)=0,\qquad \frac{\partial^{i+k}\mathbf{f}}{\partial x^i\partial t^k}(1,0)=\mathbf{0},\quad 0\le i+2k\le 4,$$

which guarantee that $\mathbf{u}\in C^{6,3}(\bar{Q})$ (the spatial partial derivatives of the solution are continuous up to sixth order and the time partial derivatives are continuous up to third order). They are an extension of the compatibility conditions for the scalar case [1]. This regularity, which is slightly higher than one would expect, is required for the analysis of the asymptotic behavior of the semidiscrete problem given below.

Singularly perturbed systems of the class (1) arise, for example, in mathematical models in fractured porous media [2]. Since the diffusion parameters ε_1 and ε_2 can take arbitrary small values, the solution of problem (1), in general, has a multiscale character [3] and therefore it is convenient to construct uniformly convergent methods (i.e., convergent for any value of the singular perturbation parameters) for its numerical approximation (see [4–6] for a general survey).

The numerical approximation of singularly perturbed parabolic systems of reaction–diffusion type has been analyzed in both the cases of two equations [7,3] and an arbitrary number of equations [8,9]. In these papers, the time variable is discretized with the Euler method on a uniform mesh and the space variable with the central difference approximation on a Shishkin mesh. Some advances have been made in constructing higher order approximations to the solution of problem (1); for example, in [10] a high order difference approximation via identity expansions (HODIE) was proposed to approximate the space variable instead of the classical central difference approximation and in [11] the Crank–Nicolson method was used for the time approximation.

The components of the discrete solution of all these numerical methods are coupled at each time level and then a high computing time is required when one considers systems with an arbitrary number of equations or multidimensional problems (see Section 4 for a further explanation). In the case of steady problems, we can cite the iterative scheme proposed by Matthews et al. [12] to solve this type of systems and the paper by Stephens and Madden [13], where the uniform convergence of the Schwarz domain decomposition method is analyzed. Up to our knowledge, additive (or splitting) methods [14,15], which are designed for a more efficient computational implementation, have not been used to approximate singularly perturbed systems. Additive schemes have a considerable interest in the case of time-dependent vector problems because they can be designed so that the components of the discrete solution are decoupled at each time level.

In this paper we consider two additive schemes define on a uniform mesh to approximate the time variable of problem (1) and the uniform convergence of the solution of the semidiscrete problem to the solution of the continuous problem is proved in Section 2 (see [16] for a detailed discussion in the case of a scalar parabolic problem). In addition, the structure of the solution of such problems is established by considering an appropriate decomposition. The analysis, which is based on an inductive argument, is given in an Appendix because it is quite technical. For the sake of completeness, the standard discretization with the Euler method is also considered in our analysis because the asymptotic behavior of its solution has not been established previously in the literature.

In Section 3 the semidiscrete problems associated to the additive schemes are discretized by using the central difference approximation on a piecewise uniform mesh of Shishkin type (see, for example, [17,3]) and the uniform convergence of the fully discrete schemes trivially follows from the papers dealing with the steady version of problem (1) (see, for example, [18,17,19,12]). We also refer to [20] for an asymptotic approach of the solution and [21] for a survey on the numerical solution of systems of singularly perturbed differential equations.

In Section 4 we give the numerical results for a system of two equations, which corroborate the order of convergence theoretically proved for the additive schemes. The additive and Euler methods are used to solve the same test problem and the computational time required to obtain the numerical approximations with these schemes and the approximated errors using the double-mesh principle are given. These results show that the additive schemes are more efficient than the standard discretization.

The additive methods proposed in this paper and the results of convergence can be generalized to systems with an arbitrary number of singularly perturbed parabolic reaction–diffusion equations if one disposes of a precise information of the asymptotic behavior of the solution of the semidiscrete problem. In [9] the structure of the solution of systems of more than two equations was established by means of an appropriate decomposition of the solution and a similar decomposition could be used for the solution of the semidiscrete problem. In Section 4 we also consider a second test problem for a system of three equations and the numerical results show that the type of additive schemes proposed in this paper are uniformly convergent and significatively more efficient than the Euler method.

In this paper we denote by $\mathbf{v}\le\mathbf{w}$ if $v_i\le w_i,\ i=1,2$, $|\mathbf{v}|=(|v_1|,|v_2|)^T$ and $\|\mathbf{f}\|_H=\max\{\|f_1\|_H,\|f_2\|_H\}$ where $\|f\|_H$ is the maximum norm of f on the closed set H. Henceforth, C denotes a generic positive constant independent of the diffusion parameters ε_1 and ε_2, and also of the discretization parameters N and M. We use $\mathbf{v}\le\mathbf{C}$ meaning that $v_1\le C,\ v_2\le C$.

In this paper we will use repeated times the two following properties:

(P1) It holds $\|g'\|_J\le\frac{2}{\mu}\|g\|_J+\mu\|g''\|_J$, where $J=[a,a+\mu]$ is an arbitrary interval with $\mu>0$ and $g\in\mathbf{C}^2(J)$ (see [22]).

(P2) If $|\Psi(x)|\le CB_\mu(x)$ and $|\Psi''(x)|\le C\mu^{-1}B_\mu(x)$, with

$$B_\mu(x)=e^{-x/\sqrt{\mu}}+e^{-(1-x)/\sqrt{\mu}},$$

then, $|\Psi'(x)|\le C\mu^{-1/2}B_\mu(x)$ (see [17,23]).