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Global existence and uniform decay rates for the semi-linear wave equation with damping and source terms

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ABSTRACT

In this paper, we consider a semi-linear wave equation with damping and source terms. Using a potential well method, we prove the existence and uniqueness of global solutions of the wave equation and investigate uniform decay rates of solutions. Moreover, an example is given to illustrate our results.

1. Introduction

In this paper, we are concerned with the existence and uniform energy decay rates for the semi-linear wave equation with damping and source terms:

$$\begin{cases} u'' - \Delta u + u' = |u|^{\alpha} u \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{on } \Gamma \times (0, +\infty), \\ u(x, 0) = u^0(x), \qquad u'(x, 0) = u^1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with boundary Γ. Let ν be the outward normal to Γ. Δ stands for the Laplacian with respect to the spatial variables and $'$ denotes the derivative with respect to time t. The damping-source interplay in the problem (1.1) arises naturally in many contexts, for instance, in classical mechanics, fluid dynamics and quantum field theory (see [1–4]). The interaction between two competitive forces, that is the damping term and the source term, makes the problem attractive from the mathematical point of view.

The problem of proving uniform decay rates for the wave equations has recently attracted a lot of attention and various results are available (some of the most important papers are those of Chen [5], Haraux [6], Komornik and Zuazua [7], Nakao [8], and Zuazua [9]). While energy decay has been extensively studied for interior sources (see for instance [10–21]), few results are available for boundary sources (cf. [2,22–27]). But there is none, to our knowledge, for the simulation result of the system having source term. On the other hand, recently, [28,29] studied the nonlinear Kirchhoff type equation on spring boundary conditions without the source term, one introduced us a concrete example and simulation results.

In this work, our present interest is to study wave equations with source term and give an example and simulation. One of our main purpose is to prove the existence of solutions by using Galerkin's method and to investigate energy decay of solutions. For the existence of solutions, first we are going to consider regular solutions and then, using the density argument, we will obtain weak solutions. However, it is not possible to consider density arguments to pass from regular solutions

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to weak solutions if one considers regular solutions where the source term is a locally Lipschitz function. To overcome this difficulty, we use a truncated function by the source term. Finally, we give an example and some numerical results to concrete our results.

This paper is organized as follows. In Section 2, we recall the notation, hypotheses and some necessary preliminaries and introduce our main results. In Section 3, we prove the existence and uniqueness of solutions for the problem (1.1) by employing Galerkin's and potential well method. In Section 4, we prove the energy decay rate by using the multiplier technique. In Section 5, we give an example and numerical simulations to illustrate our results.

2. Preliminaries

We begin this section by introducing some hypotheses and our main results. Throughout this paper, we use standard functional spaces and denote that $\|\cdot\|_p$ is the $L^p(\Omega)$ norm and $(u, v) = \int_\Omega u(x)v(x)dx$. Unless stated otherwise, the constant C is a generic constant, different in various occurrences.

(H) *Hypotheses on Ω and α.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 1$, with boundary Γ and let α be a constant satisfying the following condition:

$$0 < \alpha < \frac{2}{n-2} \quad \text{if } n \geq 3 \quad \text{and} \quad \alpha > 0 \quad \text{if } n = 1, 2.$$

In order to consider the problem (1.1), we need a potential well theory. The following remark introduces the potential well theory. Some definitions and computations can be found in [30] and references therein. For the reader's comprehension, we will repeat them here.

Remark 2.1. According to hypotheses on α, we have the imbedding

$$H_0^1(\Omega) \hookrightarrow L^{2(\alpha+1)}(\Omega) \hookrightarrow L^{\alpha+2}(\Omega).$$

We set

$$0 < K_0 := \sup_{u \in H_0^1(\Omega), u \neq 0} \left(\frac{\|u\|_{\alpha+2}}{\|\nabla u\|_2} \right) < +\infty \tag{2.1}$$

and the functional

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}, \quad u \in H_0^1(\Omega) \tag{2.2}$$

which are well defined in view of the above imbedding.

We define the positive number

$$d_0 := \inf_{u \in H_0^1(\Omega), u \neq 0} \left\{ \sup_{\lambda > 0} J(\lambda u) \right\}.$$

Setting

$$j(\lambda) = \frac{1}{2}\lambda^2 - \frac{1}{\alpha+2}K_0^{\alpha+2}\lambda^{\alpha+2}, \quad \lambda > 0, \tag{2.3}$$

then

$$\lambda_0 = \left(\frac{1}{K_0^{\alpha+2}} \right)^{\frac{1}{\alpha}}$$

is the absolute maximum point of j and $d_0 = j(\lambda_0) > 0$. Moreover, since $\lambda_0 > 0$, we have

$$1 - K_0^{\alpha+2}\lambda_0^\alpha = 0.$$

It is well known that the number d_0 is the Mountain Pass level associated to the elliptic problem

$$\begin{cases} -\Delta u = |u|^\alpha u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

In fact, $d_0 = \inf_{\kappa \in \Lambda_0} \sup_{t \in [0,1]} J(\kappa(t))$ (see [31]), where $\Lambda_0 = \{\kappa \in C([0, 1]; H_0^1(\Omega)); \kappa(0) = 0, J(\kappa(1)) < 0\}$. Furthermore, we get

$$d_0 = j(\lambda_0) = \frac{1}{2}\lambda_0^2 - \frac{1}{\alpha+2}K_0^{\alpha+2}\lambda_0^\alpha\lambda_0^2 = \frac{\alpha}{2(\alpha+2)}\lambda_0^2.$$