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Unconditionally stable numerical method for a nonlinear partial integro-differential equation

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ABSTRACT

The paper presents an unconditionally stable numerical scheme to solve a nonlinear integro-differential equation which arises in mathematical modeling of the penetration of a magnetic field into a substance, if the temperature is kept constant throughout the material. Numerical scheme comprises of the Galerkin finite element method (Jangveladze and Kiguradze, 2011) for the spatial discretization followed by an implicit finite difference scheme for the time stepping. We extended the results for stability estimates to a non homogeneous problem and derived optimal order error estimates for the semidiscretized and fully discretized equations using H_0^1 projection. Further, to show the efficiency, the proposed numerical method is demonstrated via numerical example.

1. Introduction

The process of penetration of a magnetic field into a substance generates a variable electric field causing the currents to appear that leads to the heating which further influences the resistance of the material. The process of propagation of a magnetic field into a substance and change in temperature due to Joule heating into a medium whose electric conductivity substantially depends on temperature are mathematically modeled by the following (Maxwell's) system of partial differential equations [1]:

$$\frac{\partial H}{\partial t} = -\mathrm{rot}(\nu_m \mathrm{rot} H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\mathrm{rot} H)^2, \tag{1.2}$$

where H represents the magnetic field intensity vector (H_1, H_2, H_3), θ is the temperature, c_ν and ν_m characterize the heat capacity and the electric conductivity of the material. In the case, c_ν and ν_m are the functions of temperature θ, then on integrating (1.2) with respect to time t and substituting the resulting equation in (1.1), the system obtained has the form

$$\frac{\partial H}{\partial t} = -\mathrm{rot}\left[a\left(\int_0^t |\mathrm{rot} H|^2 \, d\tau \right) \mathrm{rot} H \right], \tag{1.3}$$

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where $a = a(s)$ is defined for $s \in (0, \infty)$. For a planar magnetic field $H = (0, 0, u)$, where $u = u(x, t)$. Then, rot$H = (0, -\frac{\partial u}{\partial x}, 0)$ and the system acquires the following form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial u}{\partial x}\right)^2 d\tau\right)\frac{\partial u}{\partial x}\right]. \tag{1.4}$$

For a system (1.3), if the temperature is kept constant throughout the material, the same process of penetration of a magnetic field into a substance can be rewritten in one dimensional analogue as (transformed by Laptev [2])

$$\frac{\partial u}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 u}{\partial x^2}. \tag{1.5}$$

Eqs. (1.3) and (1.4) originated in the article [3] where the existence of a weak solution to the first boundary value problem for the one dimensional spatial version for the case $a(s) = (1 + s)$ and uniqueness results for some general cases were proved. The existence and uniqueness of (1.5) has been proved in [4]. Lions [5] presented Galerkin methods and the methods based on compactness, monotonicity, regularization and successive approximations which were used to yield the existence theorems. Existence and uniqueness theorems based on the compactness methods [5] were also proved in the articles [3,6].

In the context of the theory of nonlinear integro-differential equations and systems (1.4) and (1.5), results concerning the large time behavior, asymptotic behavior of the solutions for the different cases of $a(s)$ with different boundary conditions for the first boundary value problem and the initial boundary value problem were briefly discussed in [7–14] and the references cited therein.

For the special case $a(s) = (1+s)$, the study of the asymptotic behavior of solutions (as $t \to \infty$) and the implementation of semidiscrete and the finite difference schemes for the model (1.5), with homogeneous Dirichlet boundary conditions were carried out in [7].

During the last decades, developing the finite difference and the finite element schemes for the parabolic integro-differential equations (1.3)–(1.5) have been the focus of intensive research. Construction and investigation of these discretization schemes were discussed by the authors (see [7,15–19]) and the references therein. Recently in 2013, T. Jangveladze [19] discussed the properties of existence, uniqueness and asymptotic behavior of the solutions of nonlinear integro-differential system (1.4).

Numerical methods for partial integro-differential equations constitute an indivisible part of modern engineering and science. Starting in 1962, Douglas and Jones [20] studied the numerical solution of parabolic integro-differential equations. The authors formulated backward difference and Crank Nicholson type methods for nonlinear parabolic integro-differential equations in one space variable subject to homogeneous Dirichlet's boundary conditions. A somewhat similar hyperbolic integro-differential equations were also treated. Several authors examined various properties of both linear and nonlinear parabolic partial integro-differential equations in n space variables in papers presented at the conference "Integro-differential Evolution Equations and Applications, Vol. 10, 1985" held in Trento, Italy, in 1984 and published in the *Journal of Integral Equations*. Later in 1988, Yanik and Fairweather [21] presented a fully discrete Galerkin finite element approximation to the solutions of a certain parabolic and hyperbolic partial integro-differential equations. Study of different types of partial integro-differential equations seems to have grown exponentially in the last few decades; so that a tremendous variety of models have now been formulated, mathematically analyzed and applied to the respective areas. The breadth of the numerical analysis of various parabolic integro-differential equations is shown in the review of the literature [21–26] and the references cited therein.

In 2011, Jangveladze et al. [18] developed a conditionally stable numerical solution of (1.5) (for the special case $a(s) = 1 + s$) using the Galerkin finite element method and an explicit finite difference scheme. Recently in 2013, the work is extended to a system of integro-differential equations of the form (1.4) [19]. The authors analyzed the spatial discretization and derived error estimates for semidiscretization in the energy norm.

The goal of the present work is to obtain an unconditionally stable numerical scheme and to analyze the semidiscretize and fully discretize equations. Projection plays an important role in the derivation of optimal error estimates for the finite element approximations. The concept of projection will be seen to unify much of the analysis and the estimates are derived in an easy way.

A brief outline of this paper is as follows. In Section 2, we recall the basic definitions which are used in our analysis. In Section 3, we obtain the stability estimates for the non homogeneous problem, we also recall the weak formulation of the initial boundary value problem. In the subsections, spatial discretization of the concerned equation with the Galerkin finite element scheme is studied to derive an integro-differential equation system. H_0^1 projection of u in S^h is introduced and studied. We also analyze the semidiscrete error bounds. Implicit finite difference method is applied for a time discretization. Analogous stability estimates and the error estimates are also obtained for the fully discrete scheme. In Section 4, the discussion continues with the construction of a numerical scheme to obtain the solution of an integro-differential equation system. In the last section, we present numerical results to illustrate the efficiency and the unconditional stability of the proposed method.