Computers and Mathematics with Applications 67 (2014) 77–90

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A penalty method for a fractional order parabolic variational inequality governing American put option valuation

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ARTICLE INFO

Article history:
Received 18 May 2013
Received in revised form 5 September 2013
Accepted 21 October 2013

Keywords:
Fractional Black–Scholes operator
American option pricing
Variational inequality
Penalty method
Complementarity problem
Finite difference

ABSTRACT

A power penalty method is proposed for a parabolic variational inequality or linear complementarity problem (LCP) involving a fractional order partial derivative arising in the valuation of American options whose underlying stock prices follow a geometric Lévy process. We first approximate the LCP with a nonlinear fractional partial differential equation (fPDE) with a penalty term. We then prove that the solution to the nonlinear fPDE converges to that of the LCP in a Sobolev norm at an exponential rate depending on the parameters used in the penalty term. Numerical results are presented to demonstrate the convergence rates and usefulness of the penalty method for pricing American put options of this type.

1. Introduction

A derivative is a financial instrument whose pay-off depends on the value of another financial variable, called the underlying entity such as an asset, as well as on other financial market factors. There are two groups of derivatives: 'over-the-counter' derivatives such as swaps, which can only be traded privately and 'exchange-traded derivative contracts' which are tradable in a stock exchange market. A typical example of the latter is an option on an asset that gives its owner the right, not obligation, to buy (*call*) or sell (*put*) a certain amount of the underlying asset at an agreed price, called *strike/exercise price*, over a period of time. There are two major types of option — *European options* which can only be exercised on a specified future date (*expiry date*) and *American options* which can be exercised before or on the expiry date. Valuation of options has been a hot research topic for both mathematicians and financial engineers ever since the publication of the landmark work by Black and Scholes [1] in 1973. They showed that, in a complete market with a constant risk-free rate and without costs on transactions of stocks and bonds, the price of an option on a stock whose price follows a geometric Brownian motion with constant drift and volatility satisfies a second order parabolic partial differential equation, known nowadays as the Black–Scholes (BS) equation (or model). However, financial data show that the Gaussian shocks used in the model underestimate the probability that stock prices exhibit large jumps over small time steps. To remedy this, various models have been proposed based on two types of Levy process: the jump-diffusion and infinite activity Levy processes. In a jump-diffusion process, jumps are considered as rare events, i.e., in a given finite time interval there are only a finite number of jumps [2,3]. An infinite activity Levy process, based on the assumption that there are infinitely many jumps in a finite time interval, can capture both frequent small and rare large moves. In this work, we are concerned with the latter. It is proposed in [4] that the underlying stock price S_t of an option follows the following geometric Lévy process:

$$d(\ln S_t) = (r - v)dt + dL_t \tag{1}$$

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0898-1221/$ – see front matter
http://dx.doi.org/10.1016/j.camwa.2013.10.007

with the solution

$$S_T = S_t e^{(r-v)(T-t)+\int_t^T dL_u},$$

where T is a future known date, r is the risk-free rate, v is a convexity adjustment so that the expectation of S_T becomes $\mathbb{E}[S_T] = e^{r(T-t)}S_t$, and dL_t is the increment of a Lévy process under the equivalent martingale measure (EMM). Boyarchenko and Levendorskii [5] proposed the use of a modified Lévy-stable (LS) (Lévy-α-stable) process to model the dynamics of securities. This modification introduces a damping effect in the tails of the LS distribution, which is known as KoBoL process. Carr, Geman, Madan and Yor [6] proposed a process, known as the CGMY process, including both positive and negative jumps. In this paper, we are concerned with options based on finite moment log-stable (FMLS) processes proposed in [7].

A time-dependent random variable X_t is a Lévy process, if and only if it has independent and stationary increments with the following log-characteristic function in Lévy–Khintchine representation

$$\ln \mathbb{E}[e^{\mathrm{i}\xi X_t}] := t\Psi(\xi) = mit\xi - \frac{1}{2}\sigma^2 t\xi^2 + t\int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}\xi x} - 1 - \mathrm{i}\xi h(x))W(dx),$$

where $\mathrm{i} = \sqrt{-1}$, $m \in \mathbb{R}$ is the drift rate, $\sigma \geq 0$ is the (constant) volatility, $h(x)$ is a truncation function, W is the Lévy measure satisfying

$$\int_{\mathbb{R}} \min\{1, x^2\} W(dx) < \infty,$$

and $\Psi(\xi)$ is the characteristic exponent of the Lévy process which is a combination of a drift component, a Brownian motion component and a jump component. These three components are determined by the Lévy–Khintchine triplet (m, σ^2, W). In Lévy's process, $W(dx) = w_{LS}(x)dx$, where $w_{LS}(x)$ is the Lévy density given by

$$w_{LS}(x) = \begin{cases} Dq|x|^{-1-\alpha} & \text{for } x < 0, \\ Dpx^{-1-\alpha} & \text{for } x > 0, \end{cases}$$

for a constant $\alpha \in (0, 2]$, where $D > 0$, $p, q \in [-1, 1]$ satisfying $p + q = 1$. The characteristic exponent of the LS process is

$$\Psi_{LS}(\xi) = -\frac{\sigma^\alpha}{4\cos(\alpha\pi/2)}\left[(1-s)(\mathrm{i}\xi)^\alpha + (1+s)(-\mathrm{i}\xi)^\alpha\right] + \mathrm{i}m\xi,$$

where α and σ are respectively the stability index and scaling parameter, $s := p - q$ is the skewness parameter satisfying $-1 \leq s \leq 1$, and m is a location parameter. When $s = 1$ (resp. $s = -1$) the random variable X is maximally skewed to the left (resp. right). When $\alpha = 2$ and $s = 0$, it becomes the Gaussian case. A particular characteristic of the FMLS process is that it only exhibits downwards jumps, while upwards movements have continuous paths. The characteristic exponent of the LS process with $s = -1$, is

$$\Psi_{FMLS}(\xi) = \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)(-\mathrm{i}\xi)^\alpha,$$

where $v := \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right)$ is the convexity adjustment of the random walk.

In [4], the authors derived a fractional Black–Scholes (fBS) equation for the European option valuation based on the FMLS process and a Fourier transform. Let $V(x, t)$ be the value of a European option whose underlying stock price satisfies (1) and $\hat{V}(\xi, t)$ is the Fourier transform of $V(x, t)$. It has been shown in [4] that $\hat{V}$ satisfies

$$\frac{\partial \hat{V}(\xi, t)}{\partial t} = [r + \mathrm{i}\xi(r - v) - \Psi(-\xi)]\hat{V}(\xi, t). \tag{2}$$

Different choices of dL_t and the convexity adjustment v will result in different fPDEs from (2). The authors in [4] also derived the fPDEs under CGMY and KoBoL processes, which are both useful damped Lévy process. Because the expected value of the stock price diverges when the distribution of the random variable X_t exhibits algebraic tails, the power-law truncation does not suitable for derivative pricing, whereas for FMLS it is not an issue. Therefore, we assume the risk-neutral asset price S_t follows the FMLS process in this paper.

It has been shown in [4] that under the transformation $x = \ln S_t$, V, the Fourier inverse transform of $\hat{V}$ in (2), satisfies the following fBS equation:

$$\mathcal{L}V := -\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial x} - b[{}_{x_{\min}}D_x^\alpha V] + rV = 0 \tag{3a}$$

for $(x, t) \in (x_{\min}, x_{\max}) \times [0, T)$ with the boundary and payoff (or terminal) conditions:

$$V(x_{\min}, t) = V_0(t), \qquad V(x_{\max}, t) = V_1(t), \tag{3b}$$

$$V(x, T) = V^*(x), \tag{3c}$$