Computers and Mathematics with Applications 67 (2014) 91–105

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Stokes equations with small parameters in half plane

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A R T I C L E I N F O

Article history:
Received 16 April 2013
Received in revised form 13 August 2013
Accepted 26 October 2013

Keywords:
Stokes systems
Navier–Stokes equations
Differential equations with small parameters
Semigroups of operators
Bound ary value problems
Maximal L_p regularity

A B S T R A C T

The stationary and instationary abstract Stokes problems are studied in half plane. The equations contain an abstract operator and small parameters. The uniform well-posedness of these problems in vector-valued L^p spaces are derived.

1. Introduction

We consider the initial boundary value problem (BVP) for the following Stokes type equation with small parameter

$$\frac{\partial u}{\partial t} - \Delta_\varepsilon u + Au + \nabla\varphi = f\left(x, t\right), \qquad \operatorname{div} u = 0, \quad x \in R_+^n, \ t \in \left(0, T\right), \tag{1.1}$$

$$L_{1\varepsilon} u = \sum_{i=0}^{\nu} \varepsilon_n^{\sigma_i} \alpha_i \frac{\partial^i u}{\partial x_n^i}\left(x', 0, t\right) = 0, \quad \nu \in \{0, 1\}, \tag{1.2}$$

$$u\left(x, 0\right) = a(x), \quad x \in R_+^n, \ t \in \left(0, T\right), \tag{1.3}$$

where

$$R_+^n = \left\{x \in R^n, \ x_n > 0, \ x = \left(x', x_n\right), \ x' = \left(x_1, x_2, \ldots, x_{n-1}\right)\right\},$$

$$\sigma_i = \frac{1}{2}\left(i + \frac{1}{q}\right), \quad q \in \left(1, \infty\right), \qquad \Delta_\varepsilon u = \sum_{k=1}^{n} \varepsilon_k \frac{\partial^2 u}{\partial x_k^2},$$

A is a linear operator in a Banach space E, α_i are complex numbers and ε_k are small positive parameters. Here $f = \left(f_1\left(x, t\right), f_2\left(x, t\right), \ldots, f_n\left(x, t\right)\right)$ is a given E-valued vector function and a is an initial data such that

$$u = u_\varepsilon = \left(u_{1\varepsilon}\left(x, t\right), u_{2\varepsilon}\left(x, t\right), \ldots, u_{n\varepsilon}\left(x, t\right)\right), \qquad \varphi = \varphi\left(x, t\right)$$

are E-valued unknown functions. This problem is characterized by the presence of equations, the abstract operator A and small parameters ε_k. Parameters ε_k correspond to the inverse of very large Reynolds number Re. We prove that this problem

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0898-1221/$ – see front matter
http://dx.doi.org/10.1016/j.camwa.2013.10.016

has a unique strong maximal regular solution u on a time interval $[0, T]$ independent of ε_k. Since the Banach space E is arbitrary and A is a possible linear operator, by choosing spaces E and operators A we can obtain numerous class of Stokes type problems. For $\varepsilon_k = 1,\ E = \mathbb{C},\ A = \varkappa > 0$, the problem (1.1)–(1.3) is reduced to Stokes type problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + \varkappa u + \nabla \varphi = f(x,t), \qquad \operatorname{div} u = 0, \\ &\sum_{i=0}^{\nu} \alpha_i \frac{\partial^i u}{\partial x_n^i}\left(x', 0, t\right) = 0, \quad \nu \in \{0,1\}, \\ &u(x,0) = a(x), \quad x \in R_n^+,\ t \in (0,T), \end{aligned} \tag{1.4}$$

where $\mathbb{C}$ is the set of complex numbers. Note that, the existence of weak or strong solutions and regularity properties for the classical Stokes problems are extensively studied e.g. in [1–10]. There is an extensive literature on the solvability of the initial value problems (IVPs) for the Stokes equation (see e.g. [1,3,10] and further papers cited there). Solonnikov [8] proved that for every $f \in L^p\left(\Omega \times (0,T); R^3\right) = B(p),\ p \in (1,\infty)$ the instationary Stokes problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + \nabla \varphi = f(x,t), \qquad \operatorname{div} u = 0, \qquad u\,|_{\partial\Omega} = 0, \\ &u(x,0) = 0, \quad x \in \Omega,\ t \in (0,T) \end{aligned} \tag{1.5}$$

has a unique solution $(u, \nabla\varphi)$ so that

$$\left\| \frac{\partial u}{\partial t} \right\|_{B(p)} + \left\| \nabla^2 u \right\|_{B(p)} + \|\nabla \varphi\|_{B(p,q)} \le C \|f\|_{B(p,q)}.$$

Giga and Sohr [3] improved the result of Solonnikov for spaces with different exponents in space and time i.e., they proved that for $f \in L^p\left(0,T; (L^q(\Omega))^n\right)$ there is a unique solution $(u, \nabla\varphi)$ of the problem (1.5) so that

$$\left\| \frac{\partial u}{\partial t} \right\|_{B(p,q)} + \left\| \nabla^2 u \right\|_{B(p,q)} + \|\nabla \varphi\|_{B(p,q)} \le C \|f\|_{B(p,q)}, \tag{1.6}$$

where

$$B(p,q) = L^p\left(0,T; \left(L^q(\Omega)\right)^n\right), \quad p,q \in (1,\infty).$$

Moreover, the estimate obtained was global in time, i.e., the constant $C = C(\Omega, p, q)$ is independent of T and f. To derive the global $L^p - L^q$ estimates (1.6), Giga and Sohr used the abstract parabolic semigroup theory in *UMD* (unconditional martingale difference) spaces. We consider at first the BVP for the differential operator equation (DOE) with small parameters

$$-\Delta_\varepsilon u + (A + \lambda) u = f(x), \quad x \in R_+^n, \tag{1.7}$$

$$\sum_{i=0}^{\nu} \varepsilon_n^{\sigma_i} \alpha_i \frac{\partial^i u}{\partial x_n^i}\left(x', 0\right) = 0, \quad \nu \in \{0,1\}, \tag{1.8}$$

where A is a linear operator in a Banach space E, ε_k are positive and λ is a complex parameter. We show the separability properties of the problem (1.7)–(1.8), i.e., we prove that problem (1.7)–(1.8) has a unique solution $u \in W^{2+m,q}\left(R_+^n; E(A), E\right)$ for $f \in W^{m,q}\left(R_+^n; E\right),\ \lambda \in S_\psi$ and the following uniform coercive estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{m+2} \varepsilon_k^{\frac{i}{m+2}} |\lambda|^{1-\frac{i}{m+2}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{L^q\left(R_+^n; E\right)} + \|Au\|_{L^q\left(R_+^n; E\right)} \le C \|f\|_{W^{m,q}\left(R_+^n; E\right)},$$

where $C(q)$ is independent of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \lambda$ and f.

We then, consider the stationary abstract Stokes problem with small parameters

$$-\Delta_\varepsilon u + Au + \nabla\varphi = f(x), \quad x \in R_+^n, \qquad \operatorname{div} u = 0, \tag{1.9}$$

$$L_{1\varepsilon} u = \sum_{i=0}^{\nu} \varepsilon_n^{\sigma_i} \alpha_i \frac{\partial^i u}{\partial x_n^i}\left(x', 0\right) = 0, \quad \nu \in \{0,1\}, \tag{1.10}$$

where $f = (f_1(x), f_2(x), \ldots, f_n(x))$ is a data and $u = (u_1(x), u_2(x), \ldots, u_n(x)),\ \varphi = \varphi(x)$ is a solution. By applying the corresponding projection transformation P, the BVP (1.9)–(1.10) can be reduced to the following BVP

$$-P\Delta_\varepsilon u + Au = f(x), \quad x \in R_+^n, \tag{1.11}$$

$$\sum_{i=0}^{\nu} \varepsilon_n^{\sigma_i} \alpha_i \frac{\partial^i u}{\partial x_n^i}\left(x', 0\right) = 0, \quad \nu \in \{0,1\}. \tag{1.12}$$