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Nonnegative splittings for rectangular matrices

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ABSTRACT

The extension of the nonnegative splitting for rectangular matrices called proper nonnegative splitting is proposed first. Different convergence and comparison theorems for the proper nonnegative splittings are established. The notion of double nonnegative splitting is then generalized to rectangular matrices. Finally, different convergence and comparison results are presented for this decomposition. The case for singular square matrices is also studied.

1. Introduction

The theory of splittings for square nonsingular matrices and its relationship with the solution of system of linear equations is quite well-known. Standard iterative methods like the Jacobi, Gauss–Seidel and successive over-relaxation methods for solving a square nonsingular system of linear equations $Ax = b$, arise from different choices of real square matrices U and V, where $A = U - V$ and b is a real n-vector. The book by Varga [1] contains several splittings such as regular and weak regular splittings. A decomposition $A = U - V$ of a real square nonsingular matrix A is

(i) *regular splitting* if U^{-1} exists, $U^{-1} \geq O$ and $V \geq O$ [1],
(ii) *weak regular splitting* if U^{-1} exists, $U^{-1} \geq O$ and $U^{-1}V \geq O$ [2,1],
(iii) *nonnegative* if U^{-1} exists and $U^{-1}V \geq O$ [3],

where the comparison is entrywise and O is the null matrix. The theory of nonnegative splittings is analyzed in [4,3,5–7]. Also, Csordas and Varga [8], Elsner [9], Song [3,5], Song and Song [6], Woźnicki [10] and many others have proved various comparison results for different matrix splittings.

Berman and Plemmons [11] then extended the concept of splittings to rectangular matrices and called it as a proper splitting. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper splitting* [11] if $R(A) = R(U)$ and $N(A) = N(U)$, where $R(A)$ and $N(A)$ denote the range space and the null space of A. Linear systems of the form

$$Ax = b, \tag{1.1}$$

where A is a real square singular or real sparse or real rectangular matrix appear in many areas of mathematics. For example rectangular/singular systems arise by applying finite difference methods to partial differential equations such as the Neumann Problem and Poisson's equation. The iteration

$$x^{(i+1)} = U^{\dagger}Vx^{(i)} + U^{\dagger}b, \tag{1.2}$$

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is often employed to solve these systems, where $B^\dagger$ means the Moore–Penrose inverse of B (see next section for the definition). The scheme (1.2) is said to be *convergent* if the spectral radius of $U^\dagger V$ is less than 1, and $U^\dagger V$ is called the *iteration matrix*.

The authors of [11] showed that if $A = U - V$ is a proper splitting, then the scheme (1.2) converges to $A^\dagger b$, the least square solution of minimum norm for any initial vector x^0 if and only if the spectral radius of $U^\dagger V$ is less than 1. (See Corollary 1, [11].) *However, it is not true for any initial vector x^0. As if $x^0 = 0$ or $x^0 \in N(V)$, then the iterative sequence will not move further. Hence the initial vector x^0 should not be a zero-vector and should not lie in the null space of V.* (Interested readers can have a look at the introductory part of the article [11] for the reason and importance of choosing proper splittings.) We also remark that the scheme $Y^{(j+1)} = U^\dagger V Y^{(j)} + U^\dagger$ also converges to $A^\dagger$ under analogous conditions for a suitable initial matrix Y^0.

Very recently, the authors of [12] extended the notion of regular and weak regular splittings to rectangular matrices and the respective definitions are recalled next. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper regular splitting* if it is a proper splitting such that $U^\dagger \geq O$ and $V \geq O$. Similarly, $A = U - V$ is called *proper weak regular splitting* if it is a proper splitting such that $U^\dagger \geq 0$ and $U^\dagger V \geq O$. Note that Berman and Plemmons [11] proved a convergence theorem for these splittings without specifying the types of matrix decompositions. Using the notion of proper regular splitting, Theorem 3, [11] can be now rewritten as follows. Let $A = U - V$ be a proper regular splitting of $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger \geq O$ if and only if $\rho(U^\dagger V) < 1$. In [12], one can find comparison results for these splittings and their applications to the double splitting theory. But, in this paper, our plan is to introduce another new decomposition which is an extension of so-called nonnegative splitting for square nonsingular matrices, and is more general than the proper regular and proper weak regular splittings (see Section 3 for more discussion).

Eq. (1.1) can also be solved using double decomposition of A. (A decomposition of a real $m \times n$ matrix of the form $A = P-R-S$ is called *double decomposition*; for the real square nonsingular case it is called *double splitting* if P is nonsingular.) The idea of double splitting was first introduced by Woźnicki [13] for square nonsingular matrices. This notion was then extended by Jena et al. [12] for real $m \times n$ matrices.

Shen et al. [14], Miao and Zheng [15] and Song and Song [6] have studied convergence and comparison theorems of real square nonsingular matrices using double splittings. In particular, Song and Song [6] proved that the double splitting is convergent if and only if the single splitting is convergent for the nonnegative splittings. In this paper, we are going to extend the same result for real rectangular matrices along with a comparison result.

The central idea of this paper is to study the theory of nonnegative splittings for rectangular matrices. The organization is as follows. In Section 2, we list all relevant definitions, notation and some earlier results which we use in the paper. The main results are given in Sections 3 and 4. Section 3 introduces the generalization of nonnegative splitting to rectangular matrices, and then discusses convergence and comparison theorems for these decompositions. In Section 4, we propose the notion of double proper nonnegative splittings for real $m \times n$ matrices. Then convergence results for double proper nonnegative splittings are established. At last, we obtain a comparison theorem for two different linear systems. Section 5 discusses the group inverse analogue of a few main results mentioned in Sections 3 and 4 for square singular matrices. Finally, we end up with conclusions.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$. For a real $m\times n$ matrix A, i.e., $A \in \mathbb{R}^{m\times n}$, the matrix G satisfying the four equations known as Penrose equations: $AGA = A,\ GAG = G,\ (AG)^T = AG$ and $(GA)^T = GA$ is called the *Moore–Penrose inverse* of A (B^T denotes the transpose of B). It always exists and is unique, and is denoted by $A^\dagger$. $A \in \mathbb{R}^{m\times n}$ is said to be *semi-monotone* if $A^\dagger \geq O$. The *group inverse* of a matrix $A \in \mathbb{R}^{n\times n}$ (if it exists), denoted by $A^\#$ is the unique matrix X satisfying $A = AXA$, $X = XAX$ and $AX = XA$. Equivalently, $A^\#$ is the unique matrix X which satisfies $XAx = x$ for all $x \in R(A)$ and $Xy = 0$ for all $y \in N(A)$. The *index* of a real square matrix A is the least nonnegative integer k such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$. It is well known that $A^\#$ exists if and only if index of A is 1 (i.e., $R(A) \oplus N(A) = \mathbb{R}^n$). Let $A \in \mathbb{R}^{n\times n}$ be of index k. Then, the *Drazin inverse* of A is the unique matrix $A^D \in \mathbb{R}^{n\times n}$ which satisfies the equations $A^{k+1}A^D = A^k$, $A^DAA^D = A^D$ and $AA^D = A^DA$. $A \in \mathbb{R}^{n\times n}$ is said to be *group monotone* if $A^\#$ exists and $A^\# \geq O$. Similarly, it is called *Drazin monotone* if $A^D \geq O$. When A is a square nonsingular, then $A^\dagger = A^{-1} = A^\# = A^D$, and a semi-monotone matrix (as well as a group monotone matrix or a Drazin monotone matrix) becomes a *monotone matrix* (i.e., A^{-1} exists and $A^{-1} \geq O$). (See the book by Berman and Plemmons, [2] for more details on monotone matrices and their generalizations.) For $A, B, C \in \mathbb{R}^{m\times n}$, we say A is *nonnegative* if $A \geq O$, and $B \geq C$ if $B - C \geq O$. We denote a *nonnegative vector* x as $x \geq 0$. Let $K,\ L$ be complementary subspaces of $\mathbb{R}^p$, i.e., $K \oplus L = \mathbb{R}^p$. Then $P_{K,L}$ denotes the (not necessarily orthogonal) projection of $\mathbb{R}^p$ onto K along L. Thus $P^2_{K,L} = P_{K,L},\ R(P_{K,L}) = K$ and $N(P_{K,L}) = L$. If in addition, $K \perp L,\ P_{K,L}$ will be denoted by P_K.

A few properties of $A^\dagger$ and $A^\#$ [16] are listed here: $R(A^T) = R(A^\dagger)$; $N(A^T) = N(A^\dagger)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^T)}$; $R(A) = R(A^\#)$; $N(A) = N(A^\#)$; $AA^\# = P_{R(A),N(A)}$. In particular, if $x \in R(A^*)$ then $x = A^\dagger Ax$ and if $x \in R(A)$ then $x = A^\# Ax$. The *spectral radius* of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

The following results will be helpful to prove our main result. The first one is a part of Theorem 1, [11] and Theorem 3.1, [17] which expresses A and $A^\dagger$ in terms of U and V using the proper splitting $A = U - V$.

Theorem 2.1. *Let $A = U - V$ be a proper splitting. Then*

(a) $AA^\dagger = UU^\dagger$; $A^\dagger A = U^\dagger U$.