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# Weak solution of the equation for a fractional porous medium with a forcing term☆

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ABSTRACT

We consider the Cauchy problem with a forcing term for a fractional porous medium, which arises in statistical mechanics and heat control. The existence and uniqueness of a weak energy solution are established using implicit time discretization and $L^1$ contraction semigroup arguments.



## 1. Introduction

We consider the following Cauchy problem involving a forcing term for a fractional porous medium (FPM):

$$\begin{cases} u_t + (-\triangle)^{1/2}\left(|u|^{m-1}u\right) = f(x,t), & (x,t) \in \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $m > 0$, the forcing term is $f(x,t) \in C\left(0,\infty; L^1(\mathbb{R}^N)\right)$, and the initial data are $u_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We look for a suitable class of weak solutions for (1.1).

The fractional operator, known as the square root of the Laplacian, is defined as a pseudo-differential operator by Fourier transformation

$$\widehat{(-\triangle)^{1/2}g}(\xi) = |\xi|\hat{g}(\xi) \tag{1.2}$$

for any smooth function $g$ in the Schwartz class.

The nonlocal operator can also be expressed using the Riesz potential approach [1,2]:

$$(-\triangle)^{1/2}g(x) = C(N)\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(y)}{|x-y|^{N+1}}dy, \tag{1.3}$$

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which is an operator of integral type defined by convolution on the whole domain $\mathbb{R}^N$, where $C(N)$ is some normalization constant.

There is wide interest in studying fractional diffusion in model diffusive processes, especially for propagation of long-range diffusive interactions in porous media and infinitesimal generators of stable Lévy processes [3,4]. Fractional operators [5,6], fractional partial differential equations [7–11], and equations for porous media [12–18] have been reviewed elsewhere.

Pablo et al. considered the existence, uniqueness, and properties of Cauchy problem (1.1) without a forcing term $f \equiv 0$, and established a satisfactory theory not only for a weak solution but also for a strong solution [19]. To deal with the nonlocal operator $(-\triangle)^{1/2}$, they applied a third way to express the half Laplacian, which was introduced by Caffarelli and Silvestre [20] through the so-called Dirichlet–Neumann operator. The operator can be obtained from harmonic extension of the problem to the upper half space as the operator that maps the Dirichlet boundary condition to the Neumann one. Suppose that $v(x, y)$ is the harmonic extension of the bounded smooth function $g(x)$, written as $v = E(g)$, as the solution to

$$\begin{cases} \triangle_{x,y} v = 0, & (x, y) \in \mathbb{R}^N \times \mathbb{R}^+, \\ v(x, 0) = g(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $\triangle_{x,y}$ is the Laplacian in the variables $(x, y) \in \mathbb{R}^N \times \mathbb{R}^+$. We define the operator $T$ as $g \mapsto -\frac{\partial v}{\partial y}(x, 0)$. Applying the operator $T$ twice to $g$ yields

$$T^2 g(x) = T\left(-\frac{\partial v}{\partial y}(x, 0)\right) = u_{yy}(x, 0) = -\triangle g(x).$$

Using the fact that $T$ is a positive operator, we have

$$-\frac{\partial v}{\partial y}(x, 0) = (-\triangle)^{1/2} g(x). \tag{1.5}$$

Applying the third expression for fractional diffusion by harmonic extension to (1.1), we formulate the following problem for the extension function $w = E\left(|u|^{m-1}u\right)$:

$$\begin{cases} \triangle w = 0, & \text{for } (x, y) \in \mathbb{R}^{N+1}_+,\ t > 0, \\ \dfrac{\partial w}{\partial y} - \dfrac{\partial\left(|w|^{\frac{1}{m}-1} w\right)}{\partial t} = -f(x, t), & \text{for } x \in \mathbb{R}^N,\ y = 0,\ t > 0, \\ w(x, 0, 0) = u_0^m(x), & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

The solution $u$ of the original (1.1) can be understood as the trace of $E\left(|w|^{\frac{1}{m}-1} w\right)$ for $y = 0$. This modified problem is a quasi-stationary problem with a dynamic boundary condition. Compared to the original problem, the advantage of the modified (1.6) is that there is no nonlocal operator.

For the case $f \equiv 0$, Athanasopoulos and Caffarelli [21]showed the continuity of the bounded weak energy solution of (1.6) for $m > 1$. Pablo et al. established a systematic theory for the FPM equation without a forcing term [19]. They obtained the existence, uniqueness, and properties such as regularity, positivity, and comparison for a suitable weak solution of (1.1) for $f \equiv 0$. The existence of their solution was based on solving the modified (1.6) (for $f \equiv 0$) and an $L^1$-contraction semigroup. The results were then extended to a general FPM equation [22].

Following the ideas of Pablo et al., the main purpose of this paper is to show the existence and uniqueness of a suitable weak solution of (1.1) for a complete FPM equation. We start from the definition of a suitable weak solution of (1.1) according to the quasi-stationary problem (1.6). For convenience, we adopt the same notation as in [19], that is, $\bar{x} = (x, y)$, $\Omega = \mathbb{R}^{N+1}_+$, $\Gamma = \mathbb{R}^N \times \{0\}$, throughout the paper.

Multiplying both sides of the first equation in (1.6) by a test function $\varphi \in C_0^1(\bar{\Omega} \times [0, T))$ and integrating by parts yields

$$-\int_0^T \int_\Omega \langle \nabla w, \nabla \varphi \rangle d\bar{x} ds + \int_0^T \int_\Gamma u \frac{\partial \varphi}{\partial t} dx ds + \int_0^T \int_\Gamma f(x, s)\varphi dx ds = 0, \tag{1.7}$$

where $u$ is understood as the trace $\mathrm{Tr}\left(|w|^{\frac{1}{m}-1} w\right)$.

**Definition 1.1** (*Weak Solution*)**.** Assume that $w \in L^1\left((0, T); W^{1,1}_{\mathrm{loc}}(\Omega)\right)$, $u = \mathrm{Tr}\left(|w|^{\frac{1}{m}-1} w\right) \in L^1(\Gamma \times (0, T))$ and the identity (1.7) holds for any $\varphi \in C_0^1(\bar{\Omega} \times [0, T))$. Then the pair of functions $(u, w)$ is called a weak solution of (1.6) provided that $u(\cdot, t) \in L^1(\Gamma)$ for any $t > 0$ and $\lim_{t \to 0} u(\cdot, t) = u_0$ in $L^1(\Gamma)$.

For weak solutions the restriction conditions are relaxed and the solutions are obtained in Sobolev classes of weakly differentiable functions. However, we have to give a suitable condition to ensure the uniqueness of the problem. According to the notion for classical porous-medium equations, the concept of a weak energy solution is acceptable.