Computers and Mathematics with Applications 67 (2014) 181–198

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Coupling discontinuous Galerkin discretizations using mortar finite elements for advection–diffusion–reaction problems

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ARTICLE INFO

Article history:
Received 29 May 2013
Received in revised form 23 October 2013
Accepted 8 November 2013

Keywords:
Discontinuous Galerkin methods
Mortar finite element
Interface problem
SIPG
NIPG
IIPG

ABSTRACT

We investigate DG–DG domain decomposition coupling using mortar finite elements to approximate the solution to general second-order partial differential equations. We weakly impose an inflow boundary condition on the *inflow* part of the interface and the Dirichlet boundary condition on the *elliptic* part of the interface via Lagrange multipliers. We obtain the matching condition by imposing the continuity of the total flux through the interface and the continuity of the solution on the *elliptic* parts of the interface. The diffusion coefficient is allowed to be degenerate and the mortar interface couples efficiently the multiphysics problems. The (discrete) problem is solvable in each subdomain in terms of Lagrange multipliers and the resulting algorithm is easily parallelizable. The subdomain grids need not match and the mortar grid may be much coarser, giving a two-scale method. Convergence results in terms of the fine subdomain scale h and the coarse mortar scale H are then established. A non-overlapping parallelizable domain decomposition algorithm (Arbogast et al., 2007) reduces the coupled system to an interface mortar problem. The properties of the interface operator are discussed.

1. Introduction

Discontinuous Galerkin (DG) methods employ discontinuous piecewise polynomials to approximate the solutions of differential equations with boundary conditions and interelement continuity weakly imposed through bilinear forms. Examples of these schemes include the Bassi–Rebay method [1], the local discontinuous Galerkin (LDG) [2,3] methods, the Oden–Babuška–Baumann (OBB–DG) [4] method, and interior penalty Galerkin methods [5–7].

Even though DG solvers can be expensive due to the number of unknowns, DG methods are of particular interest for multiscale problems with several appealing properties: They are element-wise mass conservative; they support local approximations of high order; they are robust and nonoscillatory in the presence of high gradients; they are implementable on unstructured and even non-matching grids; and with appropriate meshing, they are capable of delivering exponential rates of convergence.

On the other hand, non-overlapping domain decomposition is a useful approach for spatial coupling/decoupling. A subsurface flow example is the multiblock mortar mixed finite element (MFE) method described in [8–11]. There, the governing equations hold locally on the subdomains and physically driven matching conditions are imposed on block interfaces in a numerically stable and accurate way using mortar finite element spaces. References on the mortar approach for other discretizations include [12–15] for conforming Galerkin and [16] for finite volume elements. Couplings of DG and MFE methods have been also studied in the literature. In [17], a DG–MFE coupling is introduced, which uses two Lagrange

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http://dx.doi.org/10.1016/j.camwa.2013.11.002

multipliers to impose continuity of fluxes and pressures. A method for coupling LDG and MFE is developed in [18] by choosing appropriate numerical fluxes on interface edges.

In [19], a multiscale MFE method was introduced for modeling Darcy flow. There, the continuity of the flux is imposed via a mortar finite element space on a coarse grid scale, while the equations in the coarse elements (or subdomains) are discretized on a fine grid scale. Optimal fine scale convergence is obtained by an appropriate choice of mortar grid and polynomial degree of approximation. In [20], multiscale mortar MFE–DG/DG–DG coupling methods were developed for pure diffusion problems. In [21], a multiscale mortar MFE method was also developed for nonlinear parabolic problems.

In this paper, we extend the results of [20] to a general advection–diffusion–reaction problem. We note that a DG method to advection–diffusion–reaction problems was also developed in [22]. In this paper, we develop multiscale mortar [8] DG–DG coupling methods based on four different DG formulations, the OBB–DG [4], the non-symmetric interior penalty Galerkin (NIPG) [23], the symmetric interior penalty Galerkin (SIPG) [24,7,25,26], and the incomplete interior penalty Galerkin (IIPG) [25,5,26]. In the method, the subdomain grids need not match and the mortar grid may be much coarser, giving a two-scale method. We weakly impose the boundary condition on the *inflow* part of the interface and the Dirichlet boundary condition on the *elliptic* part of the interface via Lagrange multipliers, for subdomain problems. We provide the matching condition on the interface by weakly imposing the continuity of the total flux on the interface and the continuity of the solution on the *elliptic* part of the interface via mortar finite elements. The (discrete) problem is now solvable in each subdomain in terms of Lagrange multipliers and the resulting algorithm is easily parallelizable. The diffusion coefficient is allowed to be degenerate. By using a higher order mortar approximation, we are able to compensate for the coarseness of the grid scale and maintain good (fine scale) overall accuracy. When the interface is not resolved well while the subdomain scales are fine enough, our approach also makes it easy to improve global accuracy by simply refining the local mortar grid where needed [19].

The paper is organized as follows. In the next section we introduce the model problem and formulate the weak formulation for the mortar DG and establish some notations. We also establish equivalence between the DG weak formulation and the partial differential equation. In Section 3 we introduce and analyze DG–DG mortar couplings. In particular, we establish existence and uniqueness for the discrete solution and convergence estimates. The error estimates are derived in terms of h and H, the discretization parameters for the subdomain and mortar spaces, respectively. In Section 4, we develop a parallel non-overlapping domain decomposition algorithm for the solution of the algebraic system. The solver is based on a reduction to an interface mortar problem similar to that introduced in [27,8,20] for diffusion equations. We establish coercivity of the interface operator. Finally, in Section 5, we give some concluding remarks.

2. Problem statement and notation

2.1. Model equations

Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_{12} \subset \mathbb{R}^d,\ d = 1, 2,$ or 3 be a domain with polyhedral boundary $\partial\Omega$ and $\Gamma_{12} = \partial\Omega_i \cap \partial\Omega_j$. Although for simplicity we only present the method for two subdomains, our results easily extend to geometrically nonconforming domain decompositions with finite numbers of subdomains.

We consider the following advection–diffusion–reaction problem:

$$-\nabla \cdot (\mathbf{K}(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u + c(x)u = f(x), \quad x \in \Omega, \tag{2.1}$$

where $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$ are real valued, $\mathbf{b} = \{b_i\}_{i=1}^d$ is a vector function whose entries b_i are Lipschitz continuous real valued functions on $\overline{\Omega}$ as needed in (2.4) and (3.29) given below, and $\mathbf{K} = (K_{ij})_{i,j=1}^d$ is a symmetric matrix whose entries K_{ij} are bounded, piecewise continuous real-valued functions defined on $\overline{\Omega}$, with

$$\boldsymbol{\zeta}^T \mathbf{K}(x)\boldsymbol{\zeta} \geq 0 \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^d,\ \text{a.e.}\ x \in \overline{\Omega}. \tag{2.2}$$

By the square root lemma, the matrix function $\mathbf{K}$ then admits a unique (symmetric) square root $\sqrt{\mathbf{K}}$ and it satisfies

$$\mathbf{K}\mathbf{w} \cdot \mathbf{v} = \sqrt{\mathbf{K}}\mathbf{w} \cdot \sqrt{\mathbf{K}}\,\mathbf{v} \quad \forall\, \mathbf{w},\ \mathbf{v} \in \mathbb{R}^d. \tag{2.3}$$

We also adopt the following hypothesis: There exists a positive function c_o such that

$$(c_o(x))^2 = c(x) - \frac{1}{2}\nabla \cdot \mathbf{b}(x) \quad \text{a.e.}\ x \in \Omega. \tag{2.4}$$

By $\mathbf{n}(x) = \mathbf{n}_\Omega(x)$ we denote the unit outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$. We define

$$\partial\Omega_- = \{x \in \partial\Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\} \quad \text{and} \quad \partial\Omega_+ = \{x \in \partial\Omega : \mathbf{b}(x) \cdot \mathbf{n}(x) \geq 0\}.$$

We supplement (2.1) with the boundary conditions

$$(\mathbf{b}u - \mathbf{K}\nabla u) \cdot \mathbf{n} = g_I \mathbf{b} \cdot \mathbf{n} \quad \text{on}\ \partial\Omega_-, \qquad -\mathbf{K}\nabla u \cdot \mathbf{n} = 0 \quad \text{on}\ \partial\Omega_+. \tag{2.5}$$