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A dual iterative substructuring method with a penalty term in three dimensions

Chang-Ock Lee [a], Eun-Hee Park [b,*]

[a] Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea

[b] Department of Mathematics and Center for Computation and Technology, Louisiana State University, Baton Rouge, LA, USA

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ABSTRACT

The FETI-DP method is one of the most advanced dual substructuring methods, which introduces Lagrange multipliers to enforce the pointwise matching condition on the interface. In our previous work for two dimensional problems, a dual iterative substructuring method was proposed, which is a variant of the FETI-DP method based on the way to deal with the continuity constraint on the interface. The proposed method imposes the continuity not only by the pointwise matching condition on the interface but also by using a penalty term which measures the jump across the interface. In this paper, a dual substructuring method with a penalty term is extended to three dimensional problems. A penalty term with a penalization parameter η is constructed by focusing on the geometric complexity of an interface in three dimensions caused by the coupling among adjacent subdomains. For a large η, it is shown that the condition number of the resultant dual problem is bounded by a constant independent of both subdomain size H and mesh size h. From the implementational viewpoint of the proposed method, the difference from the FETI-DP method is to solve subdomain problems which contain a penalty term with a penalization parameter η. To prevent a large penalization parameter from making subdomain problems ill-conditioned, special attention is paid to establish an optimal preconditioner with respect to a penalization parameter η. Finally, numerical results are presented.

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1. Introduction

We consider the following Poisson model problem with the homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^3$ and f is a given function in $L^2(\Omega)$. For simplicity, we assume that Ω is partitioned into two nonoverlapping subdomains $\{\Omega_i\}_{i=1}^2$ such that $\overline{\Omega} = \bigcup_{i=1}^2 \overline{\Omega}_i$. The problem (1.1) can be rewritten as

$$\min_{v_i \in H_0^1(\Omega_i, \partial\Omega)} \sum_{i=1}^{2} \left(\frac{1}{2} \int_{\Omega_i} |\nabla v_i|^2 \, dx - \int_{\Omega_i} f v_i \, dx \right) \quad \text{subject to } v_1 = v_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \tag{1.2}$$

* Corresponding author.
E-mail addresses: colee@kaist.edu (C.-O. Lee), epark2@math.lsu.edu, epark2@cct.lsu.edu (E.-H. Park).

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Here, $H_0^1(\Omega_i, \partial\Omega)$ is the usual Sobolev space defined as

$$H_0^1(\Omega_i, \partial\Omega) = \{v_i \in H^1(\Omega_i) \mid v_i = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\},$$

where $H^1(\Omega) = \{v \in L^2(\Omega) \mid \partial^\alpha v \in L^2(\Omega), |\alpha| \le 1\}$. In the domain decomposition approach based on the reformulated minimization problem (1.2) with a constraint, a key point is how to convert the constrained minimization problem into an unconstrained one. Most studies (e.g. [1–3]) for treatment of constrained minimizations started in the field of optimal control problem. The most popular methods, developed for different purposes are the Lagrangian method, the method of penalty functions, and the augmented Lagrangian method. Such various ideas have been introduced for handling constraints as the continuity across the interface in (1.2) (see [4–6]).

The FETI-DP method is the typical algorithm based on the Lagrangian method, which introduces Lagrange multipliers to enforce the continuity constraint on the interface. Many studies for the augmented Lagrangian method have been done in the frame of domain-decomposition techniques which belong to families of nonoverlapping Schwarz methods, variants of FETI method, etc. (cf. [4,7–9]). In our previous work [10] for two dimensional problems, a dual iterative substructuring method was proposed in view of the augmented Lagrangian method, which is a variant of the FETI-DP method. To the Lagrangian functional of the standard FETI-DP, a penalty term is added, which measures the jump across the interface and includes a positive penalization parameter η. In the same way as in most dual substructuring approaches, the saddle-point problem related to the augmented Lagrangian functional is reduced to the dual problem with Lagrange multipliers as unknowns. Then it is solved by the conjugate gradient method (CGM). For the preconditioned FETI-DP with the optimal Dirichlet preconditioner, it is well-known that it is numerically scalable in the sense that the condition number of the preconditioned dual problem grows asymptotically as $O(1 + \log(H/h))^2$ in two dimensions [11]. On the other hand, it was proven that the dual problem in [10] has a constant condition number independently of both of H and h even though it is not accompanied by any preconditioners.

In this paper, we extend the dual substructuring method in [10] to the three dimensional case. For this extension, there are two things to be considered: one is to construct a strong penalty term in 3-D enough to guarantee the same convergence speed as in 2-D and the other is how to treat the ill-conditioning of the subdomain problems due to a large penalization parameter. To resolve these two key issues, we need to be aware of the difference between 2-D and 3-D in the geometric complexity of the interface. An interface in 3-D includes not only faces similar to edges in 2-D but also edges which make all nodes on the interface coupled. First, it is noted that the adoption of the same penalty as suggested for two-dimensional problems in [10] gives a dual substructuring algorithm which maintains the same performance with respect to the condition number of the dual problem. However, the penalty term results in an unnecessary coupling between the functions associated with face nodes and edges nodes. Since such a coupling causes a considerable decrease on practical efficiency, we suggest a modified penalty term for the three dimensional problem aiming at reducing couplings between the functions on the interface. Next, unlike the FETI-DP method, subdomain problems containing the penalty term are solved iteratively, of which the condition number increases as a penalization parameter η increases. The same type of preconditioner as in 2-D might be a satisfactory one for the ill-conditioned problem due to a large η. But, since the preconditioner suggested in [10] contains the coupling among all nodes on the interface in 3-D, it is hardly practical in the implementational point of view. Based on such an observation, a more appropriate preconditioner for three-dimensional problems is constructed, which is not only optimal with respect to η but also more practical than one proposed in 2-D.

The rest of the paper is organized as follows. In Section 2, we introduce a dual iterative substructuring method with a penalty term. Section 3 presents algebraic condition number estimate of the resultant dual system. In Section 4, we deal with a computational issue in the implementational point of view. To remove the ill-conditioned property caused by a large penalization parameter η, an optimal preconditioner with respect to η is proposed and analyzed. In Section 5, we show numerical results. We end with concluding remarks in Section 6. To avoid the proliferation of constants, throughout the paper we will use $A \lesssim B$ and $A \gtrsim B$ to represent the statements that $A \le$ (constant)B and $A \ge$ (constant)B, where the positive constant is independent of the mesh size, the subdomain size, the number of subdomains, and the parameter η.

2. Dual iterative substructuring with a penalty term

In this section, we present a dual iterative substructuring method with a penalty term based on the augmented Lagrangian approach. We start with a minimization problem with the pointwise matching constraint on the interface. The adoption of Lagrange multipliers for dealing with the constraint yields a saddle-point problem for a Lagrangian functional. By augmenting a penalty term to the Lagrangian, we consider a slightly modified saddle-point problem which gives a dual iterative substructuring method with a penalty term.

Let $\mathcal{T}_h$ denote a quasi-uniform triangulation of Ω, where the discretization parameter h stands for the maximal mesh size of $\mathcal{T}_h$. For simplicity, we consider a triangulation into hexahedra of Ω and the standard trilinear finite element approximate solution of (1.1): find $u_h \in X_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in X_h, \tag{2.3}$$

where