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Some Tauberian theorems for the product method of Borel and Cesàro summability[☆]

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ABSTRACT

In this paper we prove several new Tauberian theorems for the product of Borel and Cesàro summability methods which improve classical Tauberian theorems for the Borel summability method.

1. Introduction and preliminaries

In 1913, the first Tauberian theorem related to the Borel summability method which gives the relation between Borel and Cesàro summability methods was given by Hardy and Littlewood [1]. Next, these well-known results extended and investigated the conditions needed for a Borel summable sequence to be convergent by Çanak and Totur [2]. In literature, a number of authors; Hardy [3], Jakimovski [4] and Rajagopal [5] investigated several sufficient conditions such that Borel summability implies (C, α).

Pati [6] proved some Tauberian theorems for the product of Abel and Cesàro summability methods, and gave some suitable conditions so that (A) (C, α) summability of $\sum_{n=0}^{\infty} a_n$ to s for some $\alpha > 0$ implies (C, α) summability of $\sum_{n=0}^{\infty} a_n$ to s. These Tauberian theorems in the sense of Pati were generalized by Çanak et al. [7], and Erdem and Çanak [8].

In what follows, we give some definitions and preliminary results in order to obtain our main theorems.

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real numbers with sequence of nth partial sums $(s_n) = (\sum_{k=0}^{n} a_k)$.

Definition 1. Let A_n^α be defined by the generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$ $(|x| < 1)$, where

$$A_0^\alpha = 1, \qquad A_n^\alpha = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)},$$

for $\alpha > -1$. If

$$s_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha} = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k \to s$$

as $n \to \infty$, we say that the sequence (s_n) is (C, α) summable to s, and it is denoted by $s_n \to s\ (C, \alpha)$.

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Particularly, for $\alpha = 1$, we observe arithmetic means of (s_n) that $s_n^1 = \sigma_n(s) = \frac{1}{n+1}\sum_{k=0}^{n} s_k$. Denote $\tau_n = na_n$ ($n = 0, 1, 2, \ldots$) and define τ_n^α as (C, α) mean of (τ_n). It is well known that if $\beta > \alpha > -1$, then $(C, \alpha) \subset (C, \beta)$ [9], and also (C, α) summability method is regular [10].

Definition 2. If the corresponding series $\sum_{n=0}^{\infty} \frac{s_n}{n!} x^n$ converges for all x and

$$e^{-x} \sum_{n=0}^{\infty} \frac{s_n}{n!} x^n \to s, \quad x \to \infty,$$

we say that the sequence (s_n) is Borel summable to s, and it is denoted by $s_n \to s\,(B)$.

Here, the Borel summability method is regular [3].

In Definition 2, if we change (s_n) by (s_n^α), then $\sum a_n$ is $(B)\,(C, \alpha)$ summable to s and represented by $s_n \to s\,(B)\,(C, \alpha)$. One can easily see that the $(B)\,(C, 0)$ summability method is the Borel summability method. In the next assertion, Szàsz [11] investigated the composition of two summability methods, as follows.

Theorem 3. *If $s_n \to s\,(B)$ as $n \to \infty$, then $s_n \to s\,(B)\,(C, \alpha)$.*

Particularly for $\alpha = 1$, if $s_n \to s\,(B)$, then $\sigma_n(s) \to s\,(B)$.

Moreover, since the Borel summability method is regular, the $(B)\,(C, \alpha)$ summability method is also regular.

Throughout this paper, we refer the symbols $u_n = o(1)$ and $u_n = O(1)$ by $u_n \to 0$ as $n \to \infty$ and (u_n) is bounded for large enough n, respectively. Now let us give the following classical Tauberian theorems.

Theorem 4 *([3, Theorem 143]). If $\sum_{n=0}^{\infty} a_n$ is Borel summable to s and $n\Delta s_n = n(s_n - s_{n-1}) = o(\sqrt{n})$, then $\sum_{n=0}^{\infty} a_n$ is convergent to s.*

Hardy and Littlewood [12] obtained the following strong result by using Theorem 4. We refer the reader to Hardy and Littlewood [13] for another proof.

Theorem 5 *([3, Theorem 156]). If $\sum_{n=0}^{\infty} a_n$ is Borel summable to s and $n\Delta s_n = O(\sqrt{n})$, then $\sum_{n=0}^{\infty} a_n$ is convergent to s.*

The main purpose of the present paper is to prove some new Tauberian theorems for the $(B)\,(C, \alpha)$ summability method which improve classical Tauberian theorems mentioned above, and to give the short proof of some classical Tauberian theorems as special cases of our results.

2. Lemmas

We begin with quoting some lemmas which are needed in proving our theorems.

Lemma 6 *([14,15]). For $\alpha > -1$, $\tau_n^\alpha = n\Delta s_n^\alpha = n(s_n^\alpha - s_{n-1}^\alpha)$.*

Lemma 7 *([15,3]). For $\alpha > -1$, $\tau_n^{\alpha+1} = (\alpha + 1)\,(s_n^\alpha - s_n^{\alpha+1})$.*

Lemma 8 *([7]). For $\alpha > -1$, $n\Delta\tau_n^{\alpha+1} = (\alpha + 1)\,(\tau_n^\alpha - \tau_n^{\alpha+1})$.*

Lemma 9 *([5]). $\sigma_n(s^\alpha) = \frac{1}{\alpha+1} s_n^{\alpha+1} + (1 - \frac{1}{\alpha+1})\sigma_n(s^{\alpha+1})$.*

The following lemma is an auxiliary result of the present paper.

Lemma 10. *If $s_n \to s(B)\,(C, \alpha)$, then $s_n \to s(B)\,(C, \alpha + 1)$.*

Proof. From the definition of the $(B)\,(C, \alpha)$ summability method, $s_n^\alpha \to s(B)$, we get $\sigma_n(s^\alpha) \to s(B)$. One can obtain from Lemma 9 that

$$\sigma_n(s^{\alpha+1}) = \frac{A_0^\alpha \sigma_0(s^\alpha) + A_1^\alpha \sigma_1(s^\alpha) + \cdots + A_n^\alpha \sigma_n(s^\alpha)}{A_0^\alpha + A_1^\alpha + \cdots + A_n^\alpha}, \tag{1}$$

where $A_0^\alpha + A_1^\alpha + \cdots + A_n^\alpha = (\frac{n}{\alpha+1} + 1)A_n^\alpha$ [3, Theorem 51]. Next, by Definition 2, we find

$$B_0(x, \sigma_n(s^\alpha)) = e^{-x} \sum_{n=0}^{\infty} \sigma_n(s^\alpha) \frac{x^n}{n!}$$

$$B_r(x, \sigma_n(s^\alpha)) = e^{-x} \sum_{n=0}^{\infty} \sigma_{n-r}(s^\alpha) \frac{x^n}{n!} = e^{-x} \int_0^y e^y B_{r-1}(y)dy \quad (r \geq 1) \text{ (see [16])}.$$

From the last equations, we get

$$\liminf_{x\to\infty} B_{r-1}(x) \geq \lim_{x\to\infty} B_r(x) \geq \limsup_{x\to\infty} B_{r-1}(x),$$